Basic Homological Algebra

May 2022

目录

1 Categories

定义 1.1. 一个 category $\mathbf{C}$ 包含如下几个部分：

- a class of objects obj $\mathbf{C}$
- sets of morphisms. 其中 morphisms 由以下方式给出：存在一个定义在 $A, B \in \mathbf{C}$ 的函数 Mor, $\mathrm{Mor}_{\mathbf{C}}(A, B)$ 被称作 the set of morphisms from A to B.
- composition:
$$\mathrm{Mor}(B, c) \times \mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C)$$
$$(g, f) \mapsto gf$$
- 一套关于 Mor 的公理系统：
  1. 每个 $\mathrm{Mor}(A, A)$ 都包含一个元素 i_A
  2. composition is associative ($(fg)h = f(gh)$)
  3. $\forall f \in \mathrm{Mor}(A, B)$, $f = fi_A = i_B f$
  4. $\mathrm{Mor}(A, B)$ 与 $\mathrm{Mor}(C, D)$ 相交 $\iff A = C,\ B = D$

注 1.2. 如果 C 在形式上不满足 4.，可以视为用三元组 (A, f, B) 代替 $f \in \mathrm{Mor}(A, B)$。

例 1. 我们可以对集合，群，拓扑空间建立范畴，此时态射分别为集合间的映射，群的同态和拓扑空间的连续映射，态射的复合即为相应映射间的复合。

例 2. 给定 $\mathbf{C}$，the opposite category $\mathbf{C}^{op}$ 定义如下:obj $\mathbf{C} =$ obj $\mathbf{C}^{op}$，$\mathrm{Mor}_{\mathbf{C}^{op}}(A, B) = \mathrm{Mor}_{\mathbf{C}}(B, A)$, composition is reversed

注 1.3. 可以看出，$\mathrm{Mor}(A, A)$ 是一个幺半群。

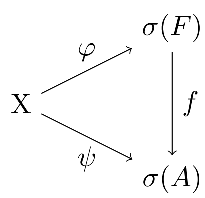
定义 1.4. $f \in \mathrm{Mor}(A, B)$ is an isomorphism $\iff \exists g \in \mathrm{Mor}(B, A)$ such that $fg = i_B,\ gf = i_A$.

注 1.5. 不难证明，g 是唯一的。

定义 1.6. $\mathbf{C}$ 被称为 concrete category，如果存在定义在 obj $\mathbf{C}$ 上的映射 σ 使得：

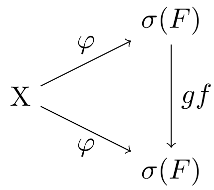
1. 如果 $A \in$ obj $\mathbf{C}$，那么 $\sigma(A)$ 是一个集合
2. 任何一个 $f \in \mathrm{Mor}(A, B)$，都导出了一个 *function f from* $\sigma(A)$ *to* $\sigma(B)$
3. composition 对应 function composition
4. i_A 导出了 $\sigma(A)$ 的恒同映射

定义 1.7. If $\mathbf{C}$ is a concrete category, $F \in$ obj $\mathbf{C}$, F is called free on a set X with a injective function $\varphi : X \to \sigma(F) \iff \forall A \in$ obj $\mathbf{C}$ and $\forall$ *set map* $\psi : X \to \sigma(A)$, there exists a unique morphism $f \in \mathrm{Mor}(F, A)$ such that $f \circ \varphi = \psi$.



命题 1.8. If F, F' are free on X with $\varphi : X \to \sigma(F)$, $\varphi' : X \to \sigma(F')$, then F and F' are isomorphic.

证明. 由 free 的定义，存在 $f \in \text{Mor}(F, F')$，$g \in \text{Mor}(F', F)$ 使得 $\varphi' = f \circ \varphi$，$\varphi = g \circ \varphi'$，则 $\varphi = gf\varphi$，此即有

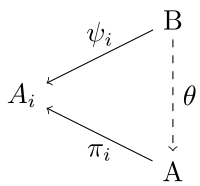


而由于 gf 的位置的函数是唯一的，且 $\varphi = i_F \circ \varphi$，从而 $i_F = gf$，同理 $fg = i_{F'}$。 □

定义 1.9. 设 $\{A_i, i \in I\}$ 是一族 obj **C** 中的对象，定义 the product of A_i, written

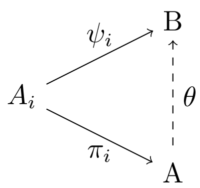
$$A = \prod_{i \in I} A_i$$

如下：A is an object, together with morphisms $\pi_i \in \text{Mor}(A, A_i)$ $\forall i \in I$, satisfying $\forall B \in \text{obj } \mathbf{C}$, $\forall \psi_i \in \text{Mor}(B, A_i)$, there is a unique $\theta \in \text{Mor}(B, A)$ such that $\theta \circ \pi_i = \psi_i$.



注 1.10. 用虚线表明待定的状态。

定义 1.11. The coproduct of A_i is an A, together with $\pi_i \in \text{Mor}(A_i, A)$, and the diagram is commutative with unique $\theta \in \text{Hom}(A, B)$:



例 3. 集合范畴的积和余积分别为集合的笛卡尔积和无交并。

定义 1.12. 群范畴下积为群的直积，余积为群的自由积。

例 4. Abel 群范畴的积是直积，余积是直和 (类似于无交并生成的最小 Abel 群)。

定义 1.13. A (covariant) functor F from **C** to **D** 是一个从 obj **C** 到 obj **D** 的函数，同时导出一个 $\text{Mor}_{\mathbf{C}}(A, B) \to \text{Mor}_{\mathbf{D}}(F(A), F(B))$，且满足

1. $F(i_A) = i_{F(A)}$
2. $F(\psi\varphi) = F(\psi)F(\varphi)$

定义 1.14. A contravariant functor from **C** to **D** is literally a covariant functor from **C** to $\mathbf{D}^{op}$.

例 5. $\mathbf{C}$ 是环的范畴，$\mathbf{D}$ 是 Abel 群的范畴，那么可以定义 F 为元素上的恒同映射。这个函子“遗忘”了 $\mathbf{C}$ 的一些性质，类似的函子称为 forgetful functor.

定义 1.15. A category $\mathbf{C}$ is called a small category if and only if obj $\mathbf{C}$ is actually a set.

定义 1.16. $\mathbf{C}$ is a subcategory of $\mathbf{D}$ if obj $\mathbf{C} \subset$ obj $\mathbf{D}$ and $\forall A, B \in$ obj $\mathbf{C}$ $\mathrm{Mor}_{\mathbf{C}}(A, B) \subset \mathrm{Mor}_{\mathbf{D}}(A, B)$。If the last set containment is always an equality, then $\mathbf{C}$ is called a full subcategory of $\mathbf{D}$.

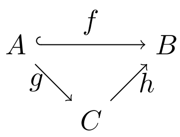
1.1 Appendix

定义 1.17. A concrete category $\mathbf{C}$ is called uniform, if for all $A \in$ obj $\mathbf{C}$, and bijective function $\varphi : \sigma(A) \to S$, there exists $B \in$ obj $\mathbf{C}$ such that $\sigma(B) = S$, and φ is an isomorphism of A with B.

定理 1.18 (Pulltab Theorem)**.** Suppose $\mathbf{C}$ is a concrete, uniform category. suppose $A, B \in$ obj $\mathbf{C}$, and $f \in \mathrm{Mor}(A, B)$. Suppose that, as a map from $\sigma(A)$ to $\sigma(B)$, f is injective. Then there exists $C \in$ obj $\mathbf{C}$, as well as $g \in \mathrm{Mor}(A, C)$, $h \in \mathrm{Mor}(C, B)$, such that $f = h \circ g$, and

(i) h is an isomorphism of C with B

(ii) $\sigma(A) \subset \sigma(C)$, and $g(x) = x, \ \forall x \in \sigma(A)$



2 Modules

2.1 Generalities

定义 2.1. 左 R 模的范畴记为 ${}_RM$，右 R 模的范畴记为 M_R，$R - S$ 双模的范畴记为 ${}_RM_S$。

命题 2.2. 给定 $A \in_R M$，$\mathrm{Hom}(A, \bullet)$ 是一个从 ${}_RM$ 到 ${}_SM$ 的协变函子，$\mathrm{Hom}(\bullet, A)$ 是一个 ${}_RM$ 到 M_S 的反变函子。

注 2.3. M_S 可以看做 ${}_{\mathbb{Z}}M_S$，${}_SM$ 可以看做 ${}_SM_{\mathbb{Z}}$。

命题 2.4. 在 Abel 群范畴（左 $\mathbb{Z}$ 模）下，$A \in M_S \cong {}_{\mathbb{Z}}M_S$，$G$ 是 Abel 群，那么 $\mathrm{Hom}(A, G)$ 可以看做 ${}_SM$ 的元素。

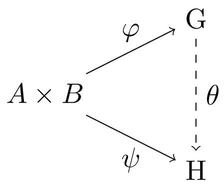
2.2 Tensor Products

定义 2.5. $A \in M_R, B \in {}_RM$, A bilinear map from $A \times B$ to an abelian group G is a map $\varphi : A \times B \to G$, satisfying $\forall a, a' \in A, \ b, b' \in B, \ r \in R$:

- $\varphi(a, b + b') = \varphi(a, b) + \varphi(a, b')$
- $\varphi(a + a', b) = \varphi(a, b) + \varphi(a', b)$

- $\varphi(ar, b) = \varphi(a, rb)$

定义 2.6. 一个序对 (G, φ) 称为 a tensor product of A, B, 如果对任意 Abel 群 H，对任意双线性映射 $\psi : A \times B \to H$，存在唯一一个 $\theta \in \mathrm{Hom}(G, H)$，使得



命题 2.7. 假定 $B \in {}_R M$，则

1. $R \otimes B \cong B$

2. 如果 I 是一个理想，IB 是 B 的子群，那么 $(R/I) \otimes B \cong B/IB$

证明. 1. 令 $\varphi(r, b) = rb$, 这是一个双线性函数, 对任意 $\psi : R \times B \to G$, 令 $\theta = \psi(1, \bullet)$, 那么 $\theta \circ \varphi = \psi$, 即 (G, φ) 是 tensor product。

注 2.8. 显然这里可以给出另一种证法: $\sum a_i \otimes b_i \mapsto \sum a_i b_i = 0 \Rightarrow \sum a_i \otimes b_i = \sum 1 \otimes a_i b_i = 1 \otimes \sum a_i b_i = 0$，从而这是单射，而满射是显然的。

这个结论对 R 的理想 I 却不成立，即不一定有 $I \otimes B \cong B$ 成立，这是因为 $1 \notin I$，从而导致证明中无法把 a_i 移动成 b_i 的系数。

2. 建立 $f : (R/I) \otimes B \to B/IB$: $f(\bar{a}, b) = \overline{ab}$，可以看出这是良定义的满同态（不依赖 $\bar{a}$ 代表元的选取）。若 $f = 0$，则 $ab \in IB$，即 $a \in I$，这表明 $\bar{a} \otimes b = 0$，故 $\mathrm{Ker} f = 0$，即 f 为单同态，从而 f 为同构。

当然我们期望给出一个由双线性函数得到的证明:

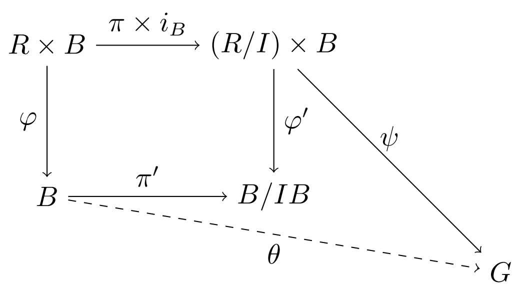
2. 定义

$$\varphi' : (R/I) \times B \to B/IB$$

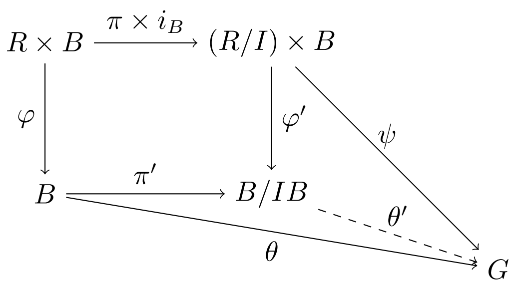
$$(r + I, b) \mapsto rb + IB$$

易证 φ' 是良定义的。

由 1. 对映射 $\pi \times i_B : R \times B \to (R/I) \times B$ 和任意双线性函数 $\psi : (R/I) \times B \to G$ 的复合，存在唯一 θ 使得 $\theta \circ \varphi = \psi \circ (\pi \times i_B)$。



对任意 $b \in B, r \in I$，有 $(\psi \circ (\pi \times i_B))(r,b) = \psi(0,b) = 0 = (\theta \circ \varphi)(r,b)$，由 1. 中 φ 的定义，有 $(\theta \circ \varphi)(r,b) = \theta(rb)$，从而 $rb \in \mathrm{Ker}\theta$，即 $I \subset \mathrm{Ker}\theta$，这表明 θ 诱导了 $\theta' : B/IB \to G$，这个映射是唯一的。



□

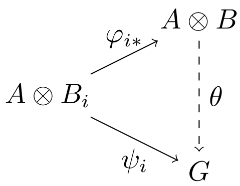
注 2.9. *可以看出 1. 的结论可以推到 R 的含幺的子环，但是对 R 的理想 I 是不成立的，即不一定有 $I \otimes B \cong B$。如 $R = \mathbb{Z}_4$，取它的理想 $I = \{0, 2\}$，取 $B = \{0, 1\} = \mathbb{Z}_2$（作为理想关于环的运算显然构成模），此时 $I \otimes B = I$，但 IB 作为 B 的子群是 $\{0\}$。*

命题 2.10. 如果 $A \in M_R$，则 $A \otimes_R$ 是一个从 ${}_RM$ 到 Abel 群的协变函子；如果 $B \in {}_RM$，那么 $\otimes_R B$ 是一个从 M_R 到 Abel 群的协变函子。

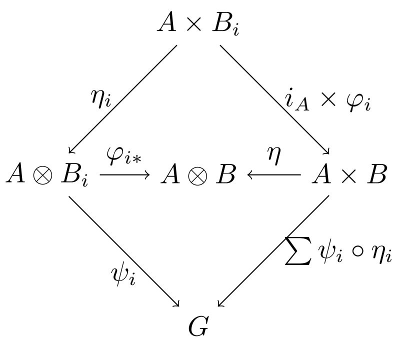
命题 2.11. 假定 $A \in M_R$，$B_i \in {}_RM$，$i \in I$，则 $A \otimes (\oplus B_i) \cong \oplus (A \otimes B_i)$，且 $a \otimes (\oplus b_i) \mapsto \oplus (a \otimes b_i)$。

证明. 即证明 $A \otimes B_i$ 在 Abel 群范畴下的余积为 $A \otimes (\oplus B_i)$。令 $B = \oplus B_i$，令 $\varphi_i \in \mathrm{Hom}(B_i, B)$ 是诱导映射，这个映射诱导出了 $\varphi_{i*} : A \otimes B_i \to A \otimes B$。

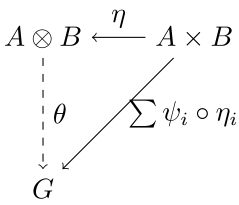
接下来只需证，对任意一组 $\psi_i : A \otimes B \to G$，存在一个 θ，使得下列图表交换：



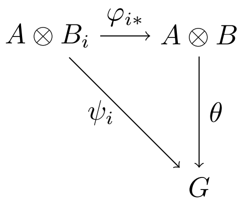
令 $\eta : A \times B \to A \otimes B$ 是双线性映射，同理有 $\eta_i : A \times B_i \to A \otimes B_i$。则 $\psi_i \circ \eta_i$ 是 $A \times B_i \to G$ 的双线性映射（群同态和双线性映射的复合是双线性的）。$\sum \psi_i \circ \eta_i$ 是良定义的（这是因为直和运算中不为 0 的分量只有有限个），这是一个 $A \times B \to G$ 的双线性映射。



由张量积的性质，此时存在唯一一个 θ 使得下图表交换：



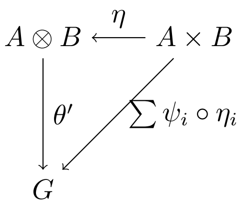
此时 $\theta \circ \eta \circ (i_A \times \varphi_i)(a, b_i) = \theta \circ \eta(a, b_i) = \sum (\psi_i \circ \eta_i)(a, b) = \sum \psi_i \circ \eta_j(a, b_j) = \psi_i \circ \eta_i(a, b_i)$，从而 $\psi_i \circ \eta_i = \theta \circ \eta \circ (i_A \times \varphi_i) = \theta \circ \varphi_{i*} \circ \eta_i$，由于 η_i 是满射，这给出下图表交换：



从而 θ 满足条件。

如果 θ 不唯一，这给出 $\theta' \circ \varphi_{i*} = \psi_i$，从而 $\theta' \circ (\eta \circ (i_A \times \varphi_i)) = \theta' \circ (\varphi_{i*} \circ \eta_i) = (\theta' \circ \varphi_{i*}) \circ \eta_i = \psi_i \circ \eta_i = \theta \circ \eta \circ (i_A \times \varphi_i)$

对每个 $(a, b) \in A \times B$，$A \times B$ 的自恒同映射由 $\sum i_A \times \varphi_i$ 给出。从而 $\theta' \circ \eta = \theta' \circ \eta \circ (\sum i_A \times \varphi_i) = \sum \theta' \circ \eta \circ i_A \times \varphi_i = \sum \theta \circ \eta \circ (i_A \times \varphi_i) = \theta \circ \eta$，这表明 θ' 也使得下图表交换



这与该图表中 θ 的唯一性相矛盾。 □

定理 2.12. 假设 $A \in M_R$，$B \in {}_RM$，G 是 Abel 群，则

$$\mathrm{Hom}_R(B.\mathrm{Hom}_{\mathbb{Z}}(A, G)) \cong \mathrm{Hom}_{\mathbb{Z}}(A \otimes B, G)$$

证明. 记 $\mathrm{Bil}(A, B; G)$ 是 $A \times B \to G$ 的所有双线性函数构成的群。根据定义，$\mathrm{Hom}_{\mathbb{Z}}(A \otimes B, G) \cong \mathrm{Bil}(A, B; G)$。

而 $\mathrm{Bil}(A, B; G) \cong \mathrm{Hom}_R(B.\mathrm{Hom}_{\mathbb{Z}}(A, G))$ 可以由以下对应得到：

$$\{maps : A \times B \to G\} \longleftrightarrow \{maps : B \to (maps : A \to G)\}$$

$$f \leftrightarrow g$$

$$f(a, b) = [g(b)]a$$

可以看出前者中的双线性映射对应后者的 R 左模同态。 □

Rotman 给出这个命题的推广版本：

定理 2.13 (Adjoint Isomorphism, First Version)**.** Given modules $A \in M_R$, $B \in {}_RM_S$, $C \in M_S$, there is a natural isomorphism:

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

定理 2.14 (Adjoint Isomorphism, Second Version)**.** Given modules $A \in {}_RM$, $B \in {}_SM_R$, $C \in {}_SM$, there is a natural isomorphism:

$$\mathrm{Hom}_S(B \otimes_R A, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

2.3 Exactness of Functors

定义 2.15. The arrows

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

will be called exact if $\mathrm{Ker}\psi = \mathrm{Im}\varphi$.

定义 2.16. The short exact sequences is like

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\pi} C \to 0$$

定义 2.17. We say that a short exact sequences splits if B is the biproduct of A and C.

注 2.18. 设 $X_1, X_2 \in \mathrm{obj}\ \boldsymbol{C}$, *the biproduct of* X_1 *and* X_2 *is* $X \in \mathrm{obj}\ \boldsymbol{C}$, *satisfying*

$$X_1 \underset{l_1}{\overset{p_1}{\leftrightarrows}} X \underset{l_2}{\overset{p_2}{\rightleftarrows}} X_2$$

$p_1 \circ l_2 = i_{X_1}$, $p_2 l_2 = i_{X_2}$, $l_1 \circ p_1 + l_2 \circ p_2 = i_X$.

定理 2.19 (5-Lemma)**.** 假定交换图表

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \downarrow{\varphi_3} & & \downarrow{\varphi_4} & & \downarrow{\varphi_5} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
\end{array}$$

上下分别正合，且

1. φ_2 和 φ_4 是同构
2. φ_1 是满射
3. φ_5 是单射

那么 φ_3 是同构。

证明. 如果 $\varphi_3(a) = 0$，那么 $\varphi_4 \circ f_3(a) = g_1 \circ \varphi_3(a) = 0$。由于 φ_4 是同构，从而 $f_3(a) = 0$，即 $a \in \mathrm{Ker}(f_3) = \mathrm{Im}(f_2)$，从而存在 $a' \in A_2$，$f_2(a') = a$。故 $g_2 \circ \varphi_2(a') = \varphi_3 \circ f_2(a') = \varphi_3(a) = 0$，从而 $\varphi_2(a') \in \mathrm{Ker}(g_2) = \mathrm{Im}(g_1)$，即存在 $b \in B_1$ 使得 $g_1(b) = \varphi_2(a')$。由于 φ_1 是满射，从而存在 a'' 使得

$\varphi_1(a'')=b$，从而 $\varphi_2\circ f_1(a'')=g_1\circ\varphi_1(a'')=g_1(b)=\varphi_2(a')$。由于 φ_2 是同构，从而 $a'=f_1(a'')$。此时 $a=f_2(a')=f_2\circ f_1(a'')=0$。即 φ_3 是单射。

如果 $b\in B_3$，则 $g_4\circ g_3(b)=0$。设 $g_3(b)=\varphi_4\circ(a)$，那么 $0=g_4\circ g_3(b)=g_4\circ\varphi_4(a)=\varphi_5\circ f_4(a)$。由于 φ_5 是单射，从而 $f_4(a)=0$，即 $a\in\mathrm{Ker}(f_4)=\mathrm{Im}(f_3)$，即存在 a' 使得 $f_3(a')=a$，从而 $g_3\circ\varphi_3(a')=\varphi_4\circ f_3(a')=\varphi_4(a)=g_3(b)$，这表明 $b-\varphi_3(a')\in\mathrm{Ker}(g_3)=\mathrm{Im}(g_2)$，即存在 $b'\in B_2$ 使得 $g_2(b')=b-\varphi_3(a')$。设 $b'=\varphi_2(a'')$，则 $\varphi_3\circ f_2(a'')=g_2\circ\varphi_2(a'')=g_2(b')=b-\varphi_3(a')$。从而 $b=\varphi_3(a'+f_2(a''))\in\mathrm{Im}(\varphi_3)$。即 φ_3 是满射。 □

定义 2.20. 定义 F 是一个从 ${}_RM$ 到 Abel 群范畴的协变函子。称 F 是正合函子，如果对任何 ${}_RM$ 中的短正合序列 $0\to A\to B\to C\to 0$ 都有 $0\to F(A)\to F(B)\to F(C)\to 0$ 正合。

如果仅有 $0\to F(A)\to F(B)\to F(C)$，则称为左正合；如果仅有 $F(A)\to F(B)\to F(C)\to 0$，则称为右正合。如果仅有 $F(A)\to F(B)\to F(C)$，那么称为半正合。

定义 2.21. 对反变函子，也可以类似定义：定义 F 是一个从 ${}_RM$ 到 Abel 群范畴的反变函子。称 F 是正合函子，如果对任何 ${}_RM$ 中的短正合序列 $0\to A\to B\to C\to 0$ 都有 $0\to F(C)\to F(B)\to F(A)\to 0$ 正合。其余定义可以类似类比。

命题 2.22.

1. 如果 $A\in {}_RM$，那么 $\mathrm{Hom}(A,\cdot)$ 和 $\mathrm{Hom}(\cdot,A)$ 是左正合的。
2. 如果 $A\in M_R$，那么 $A\otimes$ 是右正合的。

证明. 1. 设

$$0\to B\xrightarrow{\varphi}B'\xrightarrow{\psi}B''\to 0$$

是正合序列，只需证明

$$0\to\mathrm{Hom}(A,B)\xrightarrow{\varphi*}\mathrm{Hom}(A,B')\xrightarrow{\psi*}\mathrm{Hom}(A,B'')$$

是正合序列。

如果 $f\in\ker(\varphi*)$，那么 $\forall a\in A$，$\varphi(f(a))=\varphi*(f)(a)=0$，即 $f(a)\in\mathrm{Ker}(\varphi)$，即 $f(a)=0$，$\forall a\in A$，即 $f=0$。从而 $\varphi*$ 是单射，即 $0\to\mathrm{Hom}(A,B)\to\mathrm{Hom}(A,B')$ 是正合的。

对任意 $f\in\mathrm{Hom}(A,B)$，有 $\psi*(\varphi*(f))(a)=\psi(\varphi*(f)(a))=\psi(\varphi(f(a)))=0$，$\forall a\in A$，从而 $\mathrm{Im}(\varphi*)\subset\mathrm{Ker}(\psi*)$。而对任意 $g\in\mathrm{Ker}(\psi*)$，对任意 a，$\psi(g(a))=0$，从而 $\mathrm{Im}(g)\subset\mathrm{Ker}(\psi)=\mathrm{Im}\varphi$。由于 φ 是单射，从而存在定义在 $\mathrm{Im}(\varphi)$ 上的逆同态 φ^{-1}，此时令 $f=\varphi^{-1}\circ g:A\to B$，有 $\varphi*(f)(a)=\varphi(f(a))=g(a)$，即 $g\in\mathrm{Im}(\varphi*)$，从而 $\mathrm{Im}(\varphi*)=\mathrm{Ker}(\psi*)$。

$\mathrm{Hom}(\cdot,A)$ 是同理的（注意它是反变函子）。

注 2.23. 可以看到，证明中并未用到 ψ 是满射这一条件，从而可以加强为在 $0\to B\xrightarrow{\varphi}B'\xrightarrow{\psi}B''$ 下证明。

2. 设

$$0\to B\xrightarrow{\varphi}B'\xrightarrow{\psi}B''\to 0$$

是正合序列，只需证明

$$A\otimes B\xrightarrow{\varphi*}A\otimes B'\xrightarrow{\psi*}A\otimes B''\to 0$$

是正合序列。

由 1. 有

$$\mathrm{Hom}(A,\mathrm{Hom}(B,G)) \xrightarrow{\varphi*} \mathrm{Hom}(A,\mathrm{Hom}B',G) \xrightarrow{\psi*} \mathrm{Hom}(A,\mathrm{hom}(B'',G)) \to 0$$

是正合序列，G 是任意 Abel 群。

从而由2.12，有

$$\mathrm{Hom}(A\otimes B,G) \xrightarrow{\varphi*} \mathrm{Hom}(A\otimes B',G) \xrightarrow{\psi*} \mathrm{Hom}(A\otimes B'',G) \to 0$$

接下来对 1. 证明一个类似于逆命题一样的引理即可：

引理 2.24. 如果 $0 \to \mathrm{Hom}(A,B) \xrightarrow{\varphi*} \mathrm{Hom}(A,B') \xrightarrow{\psi*} \mathrm{Hom}(A,B'')$ 是正合序列，那么 $0 \to B \xrightarrow{\varphi} B' \xrightarrow{\psi*} B''$ 是正合序列。对反变函子自然反过来。

引理的证明同 1. 非常相似：

如果 $\varphi(b)=0$，那么取 $A=(b)\subset B$，那么 $\varphi(A)=0$，对 $a\in A$，此时 $\varphi*(i_A)(a)=\varphi(i_A(a))=\varphi(a)=0$，从而 $i_A=0$，这表明 $i_A(b)=b=0$。

设 $b\in B$，若 $\psi(\varphi(b))\neq 0$，那么取 $A=(b)$，此时 $\psi*(\varphi*(i_A))(b)=\psi(\varphi*(i_A)(b))=\psi(\varphi(i_A(b)))=\psi(\varphi(b))\neq 0$，矛盾。从而 $\psi(\varphi(b))=0,\forall b\in B$。这表明 $\mathrm{Im}(\varphi)\subset \mathrm{Ker}(\psi)$。

取 $A=\mathrm{Ker}(\psi)$，那么 $\psi*(i_A)=0$，从而 $i_A\in \mathrm{Im}(\varphi*)$。设 $\varphi*(f)=i_A$，那么 $\varphi*(f)(a)=i_A(a)=a,\forall a\in A$，这表明 $\varphi(f(a))=a$，即 $A\subset \mathrm{Im}(\varphi)$，从而 $\mathrm{Ker}(\psi)=\mathrm{Im}(\varphi)$。 □

2.4 Projectives, Injectives, and Flats

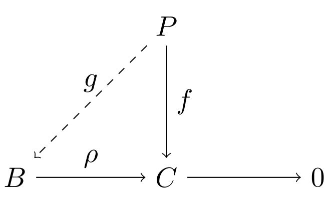
定义 2.25.

1. $A\in {}_RM$, A is projective if $\mathrm{Hom}(A,\bullet)$ is an exact functor.
2. $A\in {}_RM$, A is injective if $\mathrm{Hom}(\bullet,A)$ is an exact functor.
3. $A\in M_R$, A is flat if $A\otimes$ is an exact functor.

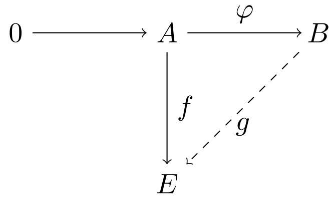
例 6. R is projective because $\mathrm{Hom}(R,B)\cong B$ via $f\mapsto f(1)$; and R is flat because $R\otimes B\cong B$.

注 2.26. 根据*2.22*:

- 模 P 是投射模当且仅当对正合列 $B\xrightarrow{\rho} C\to 0$ 及 $f\in \mathrm{Hom}_R(P,C)$，有交换图表



- 模 E 是内射模当且仅当对正合列 $0\to A\xrightarrow{\varphi} B$ 及 $f\in \mathrm{Hom}_R(A,E)$，有交换图表



注 2.27. g 不一定唯一。

命题 2.28.

- 假设 $A_i \in {}_RM$，则 $\oplus A_i$ 是投射模当且仅当每个 A_i 都是投射模；
- 假设 $A_i \in {}_RM$，则 ΠA_i 是内射模当且仅当每个 A_i 都是内射模；
- 假设 $A_i \in M_R$，则 $\oplus A_i$ 是平坦模当且仅当每个 A_i 都是平坦模。

推论 2.29. 自由模是投射模和平坦模。

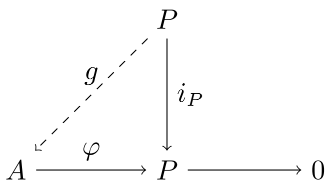
证明. 自由 R 模同构于 $\oplus R$，这里 R 不必有限。 □

推论 2.30. 每个 $A \in {}_RM$ 都是一个投射模的商映射下的像。

证明. 每个 A 都是由 A 元素生成的自由模的商模。 □

命题 2.31. 如果 $P \in {}_RM$，那么 P 是投射的当且仅当如果 $A \xrightarrow{\varphi} P$ 是满射，那么 P 是 A 的直和项。

证明. 如果 P 是投射的，那么考虑交换图表：



有 $A = \mathrm{Im}(g) \oplus \mathrm{Ker}(\phi)$ 且 $\mathrm{Im}(g) \cong P$。

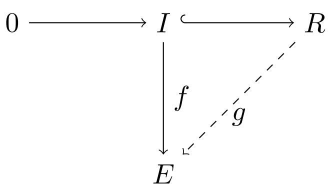
反之，考虑 A 是 P 的自由模，那么 A 是投射模，此时它的所有直和项都是投射模。 □

推论 2.32. $P \in {}_RM$ is projective if and only if every short exact sequence $0 \to A \to B \to P \to 0$ splits.

推论 2.33. 每个投射模都是一个自由模的直和项。

推论 2.34. 每个投射模 $P \in M_R$ 都是平坦模。

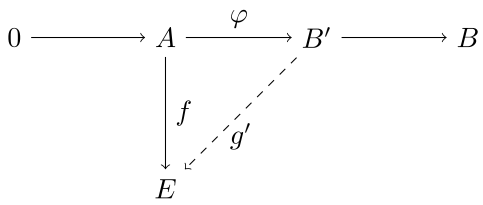
引理 2.35 (Baer)**.** 假设 $E \in {}_RM$，则 E 是内射模当且仅当对任意 R 的左理想 I 及 $f \in \mathrm{Hom}(I, E)$，存在 $g \in \mathrm{Hom}(R, E)$，使得 $g \circ i_I = f$。



证明. ⇒: 如果 E 是内射模，则由于 $0\to I\hookrightarrow R$ 是整合列，那么根据2.26，这样的 g 是存在的。

⇐: 只需对任意的正合列 $0\to A\overset{\varphi}{\to}B$，及 $f\in\mathrm{Hom}(A,E)$，证明存在 $g\in\mathrm{Hom}(B,E)$ 使得 $g\circ\varphi=f$。

利用 Zorn 引理，考虑所有的 (B',g') 使得 $\varphi(A)\subset B'\subset B$，且 $\varphi\circ g'=f$



这样的 (B',g') 是存在的，因为 $(\varphi(A),f\varphi^{-1})$ 满足条件。规定 $(B',g')\geq(B'',g'')$ 当且仅当 $B''\subset B'$ 且 $g'|_{B''}=g''$，这是一个偏序关系，且该偏序集每个链都有上界，由 Zorn 引理，该集合存在极大元 (B',g')，且不难看出极大元是唯一的。

如果 $B'\neq B$，那么取 $x_0\in B$，$x_0\notin B'$，令 $I=\mathrm{Ann}_{R/B'}(x_0)$（$X_0$ 的零化子），则 I 是 R 的左理想。

构造 $\bar{f}:I\to E$，$\bar{f}(r)=g'(rx_0)$，这个映射可以提升至 $\bar{g}:R\to E$。令 $B''=B'+Rx_0$，令 $g''(b+rx_0)=g'(b)+\bar{g}(r)$，$g''$ 是良定义的。此时 $(B'',g'')\geq(B',g')$，矛盾。 □

推论 2.36. Suppose R is a PID, and suppose $E\in{}_RM$ has the property that $rE=E$ for all $r\in R,r\neq0$, then E is injective.

证明. $I=(r)$，$f(r)\in E=rE$，设 $f(r)=ra$，那么可以取 $g(x)=xa,\forall x\in R$。 □

推论 2.37. R 是 PID，R 的分式域是内射 R 模。

注 2.38. *E is called divisible if $rE=E$ whenever r is a right nonzero divisor in R (that is, $\forall x\in R,xr=0\Rightarrow x=0$).*

注 2.39. *如果 R 是 PID，我们可以用2.36构造出一系列内射模：设 $A\in{}_RM$，记 A 生成的自由模 $F=\oplus R$，且 $A=F/K$。令 Q 是 R 的分式域，Q 是内射模。可以做嵌入 $A\cong(\oplus R)/K\hookrightarrow(\oplus Q)/K\triangleq E$，且 $\forall r\in R,r\neq0$，$rE=(\oplus(rQ))/K=(\oplus Q)/K=E$，从而 E 是内射模。*

推论 2.40. 如果 R 是 PID，那么每个 $A\in{}_RM$ 都可以嵌入一个内射模。

定理 2.41. 对一般的环 R，也可以构造内射模：假定 $A\in M_R$ 是平坦模，且 $G\in{}_{\mathbb{Z}}M$ 是内射模，那么 $\mathrm{Hom}_{\mathbb{Z}}(A,G)$ 是 ${}_RM$ 中的内射模。

证明. 假设 $0\to B\to C\to D\to0$ 正合，那么

$$0\to A\otimes B\to A\otimes C\to A\otimes D\to0$$

正合，那么

$$0\to\mathrm{Hom}(A\otimes D,G)\to\mathrm{Hom}(A\otimes C,G)\to\mathrm{Hom}(A\otimes B,G)\to0$$

正合，即

$$0\to\mathrm{Hom}(D,\mathrm{Hom}_{\mathbb{Z}}(A,G))\to\mathrm{Hom}(C,\mathrm{Hom}_{\mathbb{Z}}(A,G))\to\mathrm{Hom}(B,\mathrm{Hom}_{\mathbb{Z}}(A,G))\to0$$

正合。这代表 $\mathrm{Hom}_{\mathbb{Z}}(A.G)$ 是内射模。 □

推论 2.42. 对任意 $A \in {}_RM$，那么存在内射模 $E \in {}_RM$ 和单同态 $A \to E$。

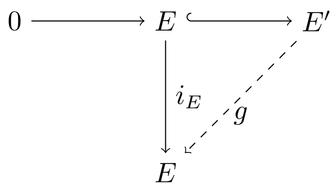
证明. 考虑 A 作为 Abel 群（即 $\mathbb{Z}$ 左模），由2.40，存在 Abel 群 G 使得 A 可以嵌入 G 且 G 是内射模。我们有

$$A \cong \mathrm{Hom}_R(R, A) \subset \mathrm{Hom}_{\mathbb{Z}}(R, A) \subset \mathrm{Hom}_{\mathbb{Z}}(R, G)$$

而 $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ 是内射模。 □

命题 2.43. E is injective if and only if E is an absolute direct summand, that is, E is a direct summand of any module having E as a submodule.

证明. 考虑 E 作为子模的嵌入 $\varphi : E \to E'$，那么 i_E 可以提升到 $g : E' \to E$，即有交换图表

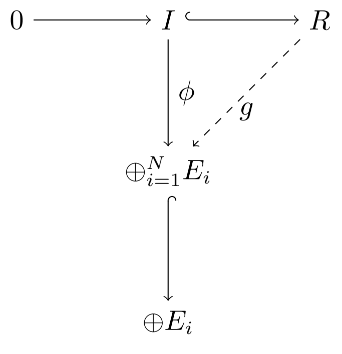


则 $\mathrm{Im}(g) \oplus \mathrm{Ker}(g) \cong E'$，而 $\mathrm{Im}(g) = E$，从而 E 是直和项。 □

定义 2.44. R is called left Noetherian if every left ideal is finitely generated.

定理 2.45. Bass-Papp] R is left Noetherian if and only if every direct sum of injectives in ${}_RM$ is injective.

证明. 令 R 是左诺特环，对 E_i 是一列内射模，和任意一个理想 $I = (a_1, \cdots, a_n)$，对任意 $\phi \in \mathrm{Hom}(I, \oplus E_i)$，由于 $\phi_j(a_k) \neq 0$ 的 j 是有限的，从而从某个 N 开始 $\phi_j(a_k) = 0, j > N, \forall k \in [1, n]$，即 $\phi(I) \subset \oplus_{i=1}^N E_i = \Pi_{i=1}^N E_i$，由2.28，$\Pi_{i=1} E_i$ 是内射模，从而有交换图表



从而此时 ϕ 被提升至 $R \to \oplus E_i$，由2.35，得出 $\oplus E_i$ 是内射模。

而如果 R 不是左诺特环，则存在严格增的左理想链 $I_1 \subset I_2 \subset \cdots$。记 $I = \bigcup_{n=1}^{\infty} I_n$，令 E_n 是包含 I/I_n 的内射模且有单射 $\varphi_n : I/I_n \to E_n$，对 $x \in I$，定义 $\varphi(x) = \oplus \varphi_n(x + I_n)$，可以看出 φ 取值在 $\oplus E_n$ 中。

如果 φ 可以提升至 g 即有 $g \circ i_I = \varphi$，记 $g_n = j_n \circ g : R \to E_n$，不难看出 g_n 是 φ_n 的提升。那么 $g_n(x) = \varphi(x + I_n) = x g_n(1), \forall x \in I - I_n$，那么 $g_n(x) \neq 0$ 从而有 $g_n(1) \neq 0$。这表明 $g(1) \notin \oplus E_i$。 □

命题 2.46. $E \in {}_RM$ is injective if and only if every short exact sequence $0 \to E \to B \to C \to 0$ splits.

定义 2.47. Let M and E be left R-modules. Then E is an essential extension of M if there is an one-to-one R-map $\alpha : M \to E$ with $S \cap \alpha(M) \neq 0$ for every nonzero submodule $S \subset E$. If also $\alpha(M) \subsetneq E$, then E is called a proper essential extension of M.

命题 2.48. A left R-module M is injective if and only if M has no proper essential extension.

证明. If M is injective but there exists a proper essential extension E of M, then M is a direct summand of E and suppose $E = M \oplus M'$. But we have $M' \cap M = 0$, a contradiction.

If M is not injective, then M is not an absolute direct summand, hence there exists $M \subset E$ such that M is not a direct summand of E. If E is not a proper essential extension of M, that is $M \cap S = 0$ for some $S \subset E$ is a submodule. By Zorn's lemma, we can make S be the maximal satisfied the property. If there exists $x \notin M + S$, then $(S + (x)) \cap M = 0$, a contradiction, so $M + S = E$, hence $E = M \oplus S$. Therefore, E is the direct summand of E, a contradiction. □

推论 2.49. Given $M \in {}_R\mathcal{M}$, the following conditions are equivalent:

1. E is a maximal essential extension of M; that is, no proper extension of E is an essential extension of M
2. E is an injective module and E is an essential extension of M
3. E is an injective module and there is no proper injective intermediate submodule E', that is, there is no injective E' with $M \subset E' \subsetneq E$.

定义 2.50. If M is a left R-module, then E containing M is an injective envelop of M, denoted by $\mathrm{Env}(M)$, if any of the equivalent conditions in 2.49 hold.

定理 2.51 (Eckmann-Schöpf). R-模同构下 $\mathrm{Env}(M)$ 保持不变。

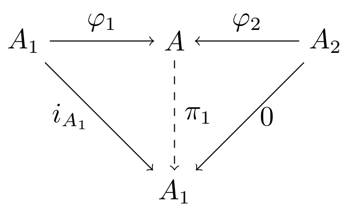
2.5 Exercises

1. Suppose only that A is a coproduct of A_1 and A_2 in ${}_R\mathcal{M}$, that is,

$$A_1 \xrightarrow{\varphi_1} A \xleftarrow{\varphi_1} A_2$$

makes A into a coproduct of A_1 and A_2 in ${}_R\mathcal{M}$. Show that there are unique $\pi_i : A \to A_i$ making A into a biproduct, using only the properties of a coproduct.

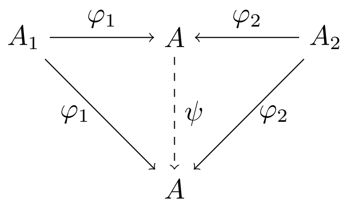
证明. π_1 arises as a filler for



The construction of π_2 is the same.

And then $i_{A_i} \circ \pi_i \circ \varphi_i = i_{A_i}$, we only need to confirm $\varphi_1 \circ i_{A_1} \circ \pi_1 + \varphi_2 \circ i_{A_2} \circ \pi_2 = i_A$.

Let $\psi = \varphi_1 \circ i_{A_1} \circ \pi_1 + \varphi_2 \circ i_{A_2} \circ \pi_2$, $\psi \circ \varphi_i = \varphi_i$, so ψ is the filler for



But i_A also makes the diagram commutative, this induces $\psi = i_A$. □

2. Suppose

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\pi} C \to 0$$

is exact, and suppose $\psi : C \to B$ satisfies $\pi \circ \psi = i_C$. Show that this sequence splits.

证明. Since $\forall b \in B$, $b = b - \psi \circ \pi(b) + \psi \circ \pi(b) \in \mathrm{Ker}(\pi) + \mathrm{Im}(\psi)$, and $\mathrm{Ker}(\pi) \cap \mathrm{Im}(\psi) = \{0\}$, we have $B = \mathrm{Ker}(\pi) \oplus \mathrm{Im}(\psi) = \mathrm{Im}(\varphi) \oplus \mathrm{Im}(\psi) \cong A \oplus \mathrm{Im}(\psi) \cong A \oplus C$. □

3. Show that $\mathrm{Hom}(A, \Pi B_i) \cong \Pi \mathrm{Hom}(A, B_i)$ and $\mathrm{Hom}(\oplus A_i, B) \cong \Pi \mathrm{Hom}(A_i, B)$

证明. Let $\pi_j : \Pi B_i \to B_j$ denote the projection, then we can induces a homomorphism

$$\Phi : \mathrm{Hom}(A, \Pi B_i) \to \Pi \mathrm{Hom}(A, B_i)$$

$$\Phi(f) = (\pi_i \circ f)_{i \in I}$$

$\Phi(f) = 0 \Rightarrow \pi_i \circ f = 0, \forall i \in I \Rightarrow f = 0$, so Φ is an isomorphism. □

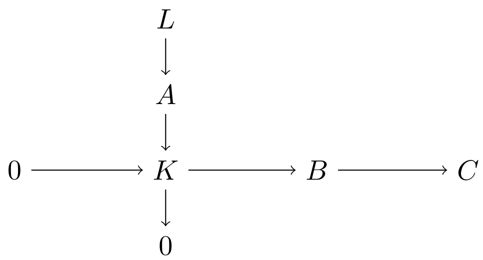
4. Suppose $B \in {}_R M$, and I is a right ideal. Show that the obvious map from $I \otimes B$ to IB is always onto. Suppose it is not one-to-one. Show that there is a finitely generated right ideal $J \subset I$ such that $J \otimes B \to JB$ is not one-to-one.

证明. Obviously we have $\sum_{j=1}^n i_j \otimes b_j \mapsto \sum_{j=1}^n i_j b_j$, so the map is onto.

If it's not a one-to-one, there exists an nonzero element $\sum_{j=1}^n i_j \otimes b_j \mapsto 0$, let $J = (i_1, \cdots, i_n)$ is finitely generated right ideal, then $J \otimes B \to JB$ is not one-to-one. □

5. Suppose F is an exact covariant functor from ${}_R M$ to Abelian groups. Show that F sends any exact sequence $A \to B \to C$ to an exact sequence $F(A) \to F(B) \to F(C)$.

证明. Let K denote the kernel of $B \to C$ and L the kernel of $A \to B$. We get the diagram



with exact row and column. Applying F and using its exactness yields the result. □

6. Show that any injective module is divisible. Also show that if a is a right zero-divisor, and if R is a submodule of E, then $1 \notin aE$ (even if E is injective).

证明. Let E denote the injective module. If $r \in R$ is a right nonzero divisor, then $0 \to Rr \xrightarrow{\varphi} R$ is an exact sequence. Hence there exists a $g_x : R \to E$ such that $g_x \circ \varphi = f_x$, $f_x(x \in E)$ denotes the map

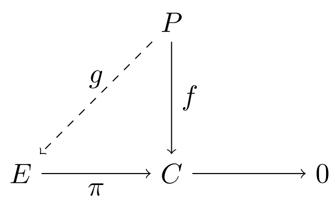
$$f_x : Rr \to E$$

$$tr \mapsto t \cdot x$$

it's well-defined because r is a right nonzero divisor. Then $x = f_x(r) = g_x(\varphi(r)) = g_x(r) = rg_x(1)$, so $\forall x \in E, x \in rE$, this shows that $E = rE$.

If a is a right zero divisor, and if R is a submodule of $E \in {}_RM$, we can assume $ra = 0, r \neq 0$, then $1 \in aE \Rightarrow r = r \cdot 1 = rae = 0$, it's a contradiction. $\square$

7. Suppose $P \in {}_RM$, and suppose a filler g exists for any diagram



when E is injective. Show that P is projective.

证明. Given $A \xrightarrow{\pi} B \to 0$, we can imbed A in an inective module E. Let j denote the map $E \to E/\mathrm{Ker}(\pi)$, then for any $f \in \mathrm{Hom}(P, E/\mathrm{Ker}(\pi))$ there exists $g \in \mathrm{Hom}(P, E)$ such that $j \circ g = f$. Since E is injective and $\varphi : A \to E$ is one-to-one, so there exists $g' \in \mathrm{Hom}(P, A)$ such that $\varphi \circ g' = g$. We have $\pi \circ g' = j|_A \circ (\varphi \circ g') = f$, so we conclude there exists $g' \in \mathrm{Hom}(P, A)$ for all $f \in \mathrm{Hom}(P, B)$. $\square$

8. Let R denote the ring of continuous functions from the real ling $\mathbb{R}$ to itself which are periodic with period π, that is, $f(x + \pi) = f(x)$ for all x. Let P denote the continuous functions from $\mathbb{R}$ to itself for which $f(x + \pi) = -f(x)$. Show that $P \oplus P \cong R \oplus R$, so that P is projective. Show also that P is not free.

证明. If $(f, g) \in R \oplus R$, then $f \sin x + g \cos x, f \sin x - g \cos x \in P$. Similarly, if $f, g \in P \oplus P$, $f \sin x + g \cos x, f \sin x - g \cos x \in R$. So $P \oplus P \cong R \oplus R$. Then P is a projective R module.

If P is free, then $P = \oplus_{i \in I} R$, then $\oplus_{i \in I}(R \oplus R) \cong R \oplus R$, it's impossible unless $P = R$. $\square$

定义 2.52. If G is a divisible Abelian group, then G will be referred to a coseparator if G contains an element of order p for every prime p.

9. Suppose G is a coseparator and $0 \neq h \in H \in \mathbf{Ab}$(*the category of Abelian groups*). Show that there is a $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(H, G)$ for which $\varphi(h) \neq 0$. (An injective coseparator in $\mathbf{Ab}$ is usually defined as an Abelian group G with this property).

证明. $0 \to (h) \to H$ is an exact sequence, obviously there exists $f \in \mathrm{Hom}((h), G)$ such that $f \neq 0$ because of coseparator, then there is a $\varphi \in \mathrm{Hom}(H, G)$ such that $g|_{(h)} = f$, hence $\varphi(h) \neq 0$. □

10.[Partial converse of 2.41.] Suppose G is a coseparator, $A \in M_R$, and suppose $\mathrm{Hom}_{\mathbb{Z}}(A, G)$ is injective. Show that A is flat.

证明. For all exact sequence

$$0 \to B \xrightarrow{\varphi} C \xrightarrow{\psi} D \to 0$$

, we can obtain another exact sequence

$$0 \to \mathrm{Hom}(D, \mathrm{Hom}(A, G)) \to \mathrm{Hom}(C, \mathrm{Hom}(A, G)) \to \mathrm{Hom}(B, \mathrm{Hom}(A, G)) \to 0$$

, then

$$0 \to \mathrm{Hom}(A \otimes D, G) \xrightarrow{\Psi} \mathrm{Hom}(A \otimes C, G) \xrightarrow{\Phi} \mathrm{Hom}(A \otimes B, G) \to 0$$

is exact.

Then we show that

$$0 \to A \otimes B \xrightarrow{\varphi*} A \otimes C \xrightarrow{\psi*} A \otimes D \to 0$$

is exact.

By using the right exactness of $A\otimes$, it suffices to show that $0 \to A \otimes B \xrightarrow{\varphi*} A \otimes C$ is exact, that is equivalent to $\varphi*$ is one-to-one.

For all nonzero $\sum a_i \otimes b_i \triangleq b \in A \otimes B$, there exists $f \in \mathrm{Hom}(A \otimes B, G)$ such that $f(b) \neq 0$ by the last exercise. Since Φ is onto, there exists $g \in \mathrm{Hom}(A \otimes C, G)$ satisfied the property $\Phi(g) = f$, i.e. $f = g \circ \varphi*$. Hence $f(b) \neq 0 \Rightarrow \varphi * (b) \neq 0$, i.e. $\mathrm{Ker}(\varphi*) = \{0\}$. □

11. Suppose $A \in M_R$. Show that A is flat if and only if $A \otimes I \to AI$ is one-to-one for every finitely generated left ideal I.

证明. If A is flat, then $0 \to A \otimes I \to A \otimes R \cong A$ is exact, and we have $\mathrm{Im}(I \to A) = AI$, then $A \otimes I \to AI$ is one-to-one.

If $A \otimes I \to AI$ is one-to-one for any finitely generated left ideal I, since it's onto, then we obtain a isomorphism $A \otimes I \cong AI$, and we can get rid of the requirement "finitely generated" by the exercise 4.

For any $f \in \mathrm{Hom}(I, \mathrm{Hom}(A, G)) \cong \mathrm{Hom}(A \otimes I, G) \cong \mathrm{Hom}(AI, G)$($G$ is a coseparator), since G is injective and $0 \to AI \hookrightarrow A$ is exact, there exists $g \in \mathrm{Hom}(A, G) \cong \mathrm{Hom}(A \otimes R, G) \cong \mathrm{Hom}(R, \mathrm{Hom}(A, G))$ such that $g|_I = f$, then $\mathrm{Hom}(A, G)$ is injective by Baer's theorem. Hence A is flat by the last exercise. □

12. Suppose R is a PID, Show that A is flat if and only if A is torsion free; that is $ar = 0 \Rightarrow a = 0\ or\ r = 0\ for\ a \in A,\ r \in R$. Hence, show $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

证明. Using the last exercise, we conclude

$$A\ is\ flat\ \Leftrightarrow\ \forall r \in R,\ A \otimes (r) \cong A(r)\ \Leftrightarrow\ (\sum a_j r_j = 0 \Rightarrow \sum a_j \otimes r_j = 0)$$

If A is flat, we have $ar = 0 \Leftrightarrow a \otimes r = 0 \Leftrightarrow a = 0 \text{ or } r = 0$.

If A is torsion free, let $\sum_{i=1}^{n} a_j r_j = \sum (a_j t_j) r = 0$. We have $\sum_{i=1}^{n} a_j \otimes r_j = \sum a_j \otimes t_j r = \sum a_j t_j \otimes r = (\sum a_j t_j) \otimes r = 0$. □

注 2.53. 注意把最后一步和2.8的第二段作比较，这里可以做到系数的转移而那里不可以。

13. Suppose R and S are rings, $A \in M_R$, $B \in {}_R M_S$, and $C \in {}_S M$. Then $A \otimes_R B \in M_S$ and $B \otimes_S C \in {}_R M$. Show that $A \otimes_R (B \otimes_S C) \cong (A \otimes_R B) \otimes_S C$.

证明. We have the natural map $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$. □

14. Suppose we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{j} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & A' & \xrightarrow{j'} & B' & \xrightarrow{\pi'} & C' & \longrightarrow & 0
\end{array}$$

in ${}_R M$ with exact rows. Prove that:

a) If η and ϕ are one-to-one, then so is ψ.

b) If η and ϕ are onto, then so is ψ.

证明. a) If $\psi(b) = 0$, then $\phi(\pi(b)) = \pi'(\psi(b)) = 0$. Since ϕ is one-to-one, we have $\pi(b) = 0$, then $b \in \mathrm{Ker}(\pi) = \mathrm{Im}(j)$. Let $j(a) = b$, then $j'(\eta(a)) = \psi(j(a)) = 0$, then $a = 0$ because η and j' are one-to-one. Hence $b = j(a) = 0$. Hence ψ is one-to-one.

b) For any $b' \in B'$, there exists $c \in C$ such that $\phi(c) = \eta'(b')$ because ϕ is onto. Since π is onto, then $c = \pi(b)$ for some $b \in B$. Hence $\eta'(\psi(b)) = \phi(\pi(b)) = \eta'(b')$, i.e. $b' - \psi(b) \in \mathrm{Ker}(\eta') = \mathrm{Im}(j')$. Hence $b' - \psi(b) = j'(a')$ for some $a' \in A'$, and $a' = \eta(a)$ for some $a \in A$ because η is onto. Since $\psi(j(a)) = j'(\eta(a)) = b' - \psi(b)$, we obtain $b' = \psi(j(a) + \psi(b)) \in \mathrm{Im}(\psi)$, therefore ψ is onto. □

15. Suppose $A \in_S M_R$, $B \in {}_R M$ and $C \in {}_S M$. Then $\mathrm{Hom}_S(A, C)$ becomes a left R-module, and $A \otimes_R B$ becomes a left S-module. Prove that $\mathrm{Hom}_S(A \otimes B, C) \cong \mathrm{Hom}(B, \mathrm{Hom}(A, C))$.

证明. The proof is similar to 2.12. □

16. Suppose R is a PID, and a is a nonzero non-unit in R. Show that R/Ra is an injective module over itself.

证明. For any $I \subset R/Ra$, I is principal ideal, let $I = (\bar{b})$. Since $a \in (b)$, we have $a = bt$ for some $t \in R$.

For any $f \in \mathrm{Hom}_{R/Ra}(I, R/Ra)$, we have $\bar{t} f(\bar{b}) = f(\bar{t}\bar{b}) = 0$. Let $\bar{c}$ denote $f(\bar{b})$, we obtain $tc \in Ra$, i.e. $tc = as = tbs$ for some $s \in R$. Hence we have $c = bs$. Now we can extend f to $g : \bar{1} \mapsto_{R/Ra} \bar{s}$, $g(\bar{b}\bar{d}) = \bar{d}\bar{b} g(1) = \bar{d}\bar{c}$, so $g|_I = f$.

Then R/Ra is injective by Baer's Theorem. □

2.6 Something about Flat Modules

Exercise 11 gives us

定理 2.54. Suppose $A \in M_R$. A is flat if and only if $A \otimes I \to AI$ is one-to-one for every finitely generated left ideal I.

命题 2.55. Let $0 \to K \to F \xrightarrow{\varphi} A \to 0$ be an exact sequence of right R-modulkes in which F is flat. Then A is a flat module if and only if $K \cap FI = KI$ for every finitely generated left ideal I.

证明. We have $K \otimes I \to F \otimes I \to A \otimes I \to 0$ is exact since $\otimes I$ is right exact.

We can define $\gamma : A \otimes I \to FI/KI$, given by $\varphi(f) \otimes i \mapsto fi + KI$, where $f \in F$, $i \in I$. The homomorphism is well-defined, if not there exists $\sum \varphi(f) \otimes i = \sum \varphi(f') \otimes i'$ gives $\sum f'i' + KI \neq \sum fi + KI$, hence there exists $\sum \varphi(f) \otimes i = 0$ but $\sum fi \notin KI$, we have $\sum f \otimes i \in \mathrm{Ker}(\varphi \otimes i_I) = \mathrm{Im}(K \otimes I \to F \otimes I)$, set $\sum k \otimes i \mapsto \sum f \otimes i$, hence $\sum ki = \sum fi \in KI$, a contradiction.

$$\begin{array}{ccccccccc}
K \otimes I & \longrightarrow & F \otimes I & \xrightarrow{\varphi \otimes i_I} & A \otimes I & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow \theta & & \downarrow \theta' & & \downarrow \gamma & & \downarrow & & \downarrow \\
KI & \longrightarrow & FI & \longrightarrow & FI/KI & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Since θ is onto, and θ' is isomorphism, 5-lemma gives us γ is isomorphism.

Suppose $\sigma : FI/KI \to FI/K \cap FI$ via $x + KI \to x + K \cap FI$, we have $\mathrm{Ker}(\sigma) = K \cap FI/KI$. Thus $A \otimes I/\ker(\sigma) \cong FI/K \cap FI$. But $\varphi(FI) = AI$ infers that $FI/\ker = AI$, and obviously the Ker $= K \cap FI$ by the exactness, hence $A \otimes I/\mathrm{Ker}(\sigma) \cong AI$. Then A is flat if and only if σ is isomorphism if and only if $FI \cap K = KI$. □

引理 2.56. Let $0 \to K \to F \to A \to 0$ be an exact sequence of right R-modules, where F is free with basis $\{x_j : j \in J\}$. For each $v \in F$. define $I(v)$ is the ideal by the coordinates of v, that is, if $v = \sum_{i=1}^{n} x_{j_i} r_i \in F$, $r_i \in R$ then $I(v) = (r_1, \cdots, r_t) \subset R$. Then A is flat if and only if $v \in KI(v)$ for every $v \in K$.

证明. A is flat if and only if $K \cap FI(v) = KI(v)$.

If A is flat, then $v \in K \cap FI(v) = KI(v)$.

If $v \in KI(v)$, for any left ideal I, let $v \in K \cap FI$, then $I(v) \subset I$, hence $K \cap FI \subset KI$. Hence $K \cap FI = KI$. □

定理 2.57 (Villamayor)**.** Let $0 \to K \to F \to A \to 0$ be exact, where F is free. The following statements are equivalent:

1. A is flat
2. For every $v \in K$, there is an R-map $\theta : F \to K$ with $\theta(v) == v$
3. For every $v_1, \cdots, v_n \in K$, there is an R-map $\theta : F \to K$ with $\theta(v_i) = v_i$ for all i

2.7 Purity

定义 2.58. An exact sequence $0 \to B' \to B \to B'' \to 0$ of left R-modules is pure exact if, for every right R-module A, we have exactness of $0 \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0$.

定理 2.59. A left R-module B'' is flat if and only if every exact sequence $0 \to B' \to B \to B'' \to 0$ of left B-modules is pure exact.

证明. 这个证明用后面的 Tor 更容易，由于 $\mathrm{Tor}_1(A, B'') \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0$ 是正合序列，从而成立。 □

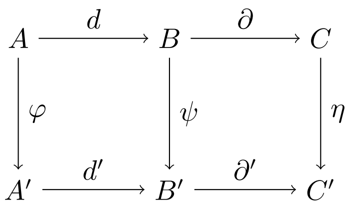
3 Ext and Tor

3.1 Complexes and Projective Resolutions

定义 3.1. For a sequence $A \xrightarrow{d} B \xrightarrow{\partial} C$, it is called a complex if $\partial \circ d = 0$.

定义 3.2. The homology of the complex is defined to be the quotient $\mathrm{Ker}(\partial)/\mathrm{Im}(d)$.

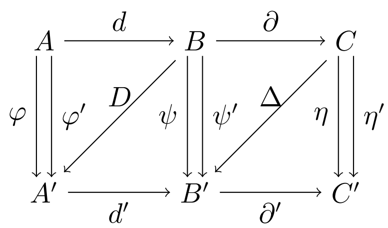
定义 3.3. Suppose



commutes, with rows are complexes. Set $H = \mathrm{Ker}(\partial)/\mathrm{Im}(d)$, $H' = \mathrm{Ker}(\partial')/\mathrm{Im}(d')$. We can now define a homomorphism from H to H' via $\psi*(x + \mathrm{Im}(d)) = \psi(x) + \mathrm{Im}(d')$. This $\psi*$ is well-defined.

定义 3.4. If there exists another φ', ψ', η' yields $\psi*'$, a homotopy is a pair of maps $D : B \to A'$ and $\Delta : C \to B'$ satisfying $\psi - \psi' = d' \circ D + \Delta \circ \partial$.

We have the diagram (obviously noncommutative)



引进同伦是因为如下性质：

命题 3.5. If a homotopy exists, then $\psi* = \psi*'$, since $\psi(x) + \mathrm{Im}(d) = \psi'(x) + d' \circ D(x) + \Delta \circ \partial(x) + \mathrm{Im}(d') = \psi'(x) + d' \circ D(x) + \mathrm{Im}(d') = \psi'(x) + \mathrm{Im}(d')$ when $x \in \mathrm{Ker}(\partial)$.

定义 3.6. Suppose $B \in {}_RM$, a projective resolution of B, denoted $< P_n, d_n >$, is an exact sequence of R-modules

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} B \to 0$$

going off to infinity to the left, in which all P_n are projective.

命题 3.7. Any left R-module has a projective resolution.

证明. 由2.30，可以取投射模 P_{n+1} 使得 $d_{n+1} : P_{n+1} \to \mathrm{Ker}(d_n)$ 是满射。 □

命题 3.8. Suppose $B, B' \in {}_RM$, and $\varphi \in \mathrm{Hom}(B.B')$. Suppose $< P_n, d_n >$ is a projective resolution of B, and $< P'_n, d'_n >$ is a projective resolution of B'. Then there exists filler $\varphi_n \in \mathrm{Hom}(P_n, P'_n)$ making

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\pi} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_n} & & & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & & \\
\cdots & \longrightarrow & P'_{n+1} & \xrightarrow{d'_{n+1}} & P'_n & \xrightarrow{d'_n} & \cdots & \longrightarrow & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\pi'} & B' & \longrightarrow & 0
\end{array}$$

commutative. Further, if $\varphi'_n \in \mathrm{Hom}(P_n, P'_n)$ also serve as fillers, then φ_n and φ'_n are homotopic, that is, there exists $D_n : P_n \to P'_{n+1}$ (with $D_{-1} = 0$) such that $\varphi_n - \varphi'_n = d'_{n+1} \circ D_n + D_{n-1} \circ d_n$.

证明. If $\varphi_0, \cdots, \varphi_n$ has been defined, note that $d'_n \circ \varphi_n \circ d_{n+1} = \varphi_{n-1} \circ d_n \circ d_{n+1} = 0$, we have $\mathrm{Im}(\varphi_n \circ d_{n+1}) \subset \mathrm{Ker}(d'_n) = \mathrm{Im}(d'_{n+1})$.

Since P_{n+1} is projective, then there exists a filler φ_{n+1} for

$$\begin{array}{ccccc}
 & & P_{n+1} & & \\
 & \swarrow{\scriptstyle \varphi_{n+1}} & \downarrow{\scriptstyle \varphi_n \circ d_{n+1}} & & \\
P'_{n+1} & \xrightarrow[d'_{n+1}]{} & \mathrm{Im}(d'_{n+1}) & \longrightarrow & 0
\end{array}$$

It remains to show any two fillers are homotopic.

Note that $\pi' \circ \varphi_0 = \varphi \circ \pi = \pi' \circ \varphi'_0$, we have $\varphi_0 - \varphi'_0$ take values in $\mathrm{Ker}(\pi') = \mathrm{Im}(d'_1)$.

Let D_0 be the filler for

$$\begin{array}{ccccc}
 & & P_0 & & \\
 & \swarrow{\scriptstyle D_0} & \downarrow{\scriptstyle \varphi_0 - \varphi'_0} & & \\
P'_1 & \xrightarrow[d'_1]{} & \mathrm{Im}(d'_1) & \longrightarrow & 0
\end{array}$$

If $D_0, \cdots, D_n$ have been defined, and we have $\varphi_n - \varphi'_n = d'_{n+1} D_n + D_{n-1} d_n$, so that

$$d'_{n+1} \circ (\varphi_{n+1} - \varphi'_{n+1} - D_n \circ d_{n+1}) = (\varphi_n - \varphi'_n - d'_{n+1} \circ D_n) \circ d_{n+1} = D_{n-1} \circ d_n \circ d_{n+1} = 0$$

Then $\mathrm{Im}(\varphi_{n+1} - \varphi'_{n+1} - D_n \circ d_{n+1}) \subset \mathrm{Ker}(d'_{n+1}) = \mathrm{Im}(d'_{n+2})$, we can denote D_n as a filler for

$$\begin{array}{ccccc} & & P_{n+1} & & \\ & \overset{D_{n+1}}{\swarrow} & \downarrow{\scriptstyle \varphi_{n+1} - \varphi'_{n+1} - D_n \circ d_{n+1}} & & \\ P'_{n+2} & \xrightarrow[d'_{n+2}]{} & \mathrm{Im}(d'_{n+2}) & \longrightarrow & 0 \end{array}$$

□

定义 3.9. Let $A \in {}_RM$. For any projective resolution of B, we have a complex sequence

$$\cdots \to A \otimes P_{n+1} \xrightarrow{i_A \otimes d_{n+1}} A \otimes P_n \xrightarrow{i_A \otimes d_n} \cdots \to A \otimes P_1 \xrightarrow{i_A \otimes d_1} A \otimes P_0 \xrightarrow{A \otimes d_0} 0$$

Let $\mathrm{Tor}_n(A, B)$ denotes the nth homology of this complex, i.e. $\mathrm{Ker}(A \otimes d_n)/\mathrm{Im}(i_A \otimes d_{n+1})$.

注 3.10. 这里 d_0 不是 π，而是 0。可以看到这里删掉了 $A \otimes B$。

命题 3.11. Up to isomorphism, the homology is independent of the choice of projective resolution.

证明. Using 3.8 twice, we have

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} \cdots \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle i_B} & & \\ \cdots & \longrightarrow & P'_{n+1} & \xrightarrow{d'_{n+1}} & P'_n & \xrightarrow{d'_n} \cdots \longrightarrow & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\pi'} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_{n+1}} & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle i_B} & & \\ \cdots & \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} \cdots \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\pi} & B & \longrightarrow & 0 \end{array}$$

Then $\psi_n \circ \varphi_n$ is homotopic to i_{P_n}, so the homomorphism between the homological groups $(i_A \otimes \varphi_n) * \circ (i_A \otimes \psi_n) * = identity$. Then the nth homological groups are isomorphism. □

命题 3.12. $\mathrm{Tor}_n(A, \bullet)$ is a covariant functor from ${}_RM$ to **Ab**. Also this functor is addictive.

证明. 事实上，对 $\varphi \in \mathrm{Hom}(B, B')$，诱导了 $A \otimes \varphi_n \in \mathrm{Hom}(\mathrm{Tor}_(A, B), \mathrm{Tor}_n(A, B'))$. □

命题 3.13. $\mathrm{Tor}_n(\bullet, B)$ 也是一个加性函子，事实上，对 $f \in \mathrm{Hom}(A, A')$，我们有交换图表

$$\begin{array}{ccccccccc} \cdots \longrightarrow & A \otimes P_{n+1} & \xrightarrow{i_A \otimes d_{n+1}} & A \otimes P_n & \xrightarrow{i_A \otimes d_n} \cdots \longrightarrow & A \otimes P_1 & \xrightarrow{i_A \otimes d_1} & A \otimes P_0 & \xrightarrow{i_A \otimes d_0} 0 \\ & \downarrow{\scriptstyle f \otimes i_{P_{n+1}}} & & \downarrow{\scriptstyle f \otimes i_{P_n}} & & \downarrow{\scriptstyle f \otimes i_{P_1}} & & \downarrow{\scriptstyle f \otimes i_{P_0}} & \\ \cdots \longrightarrow & A' \otimes P_{n+1} & \xrightarrow{i_{A'} \otimes d_{n+1}} & A' \otimes P_n & \xrightarrow{i_{A'} \otimes d_n} \cdots \longrightarrow & A' \otimes P_1 & \xrightarrow{i_{A'} \otimes d_1} & A' \otimes P_0 & \xrightarrow{i_{A'} \otimes d_0} 0 \end{array}$$

若记 f 诱导的 $\mathrm{Tor}_n(A, B)$ 到 $\mathrm{Tor}_n(A', B)$ 的同态为 $\mathrm{Tor}_n(f, B)$（依前面的记号应为 $(f \otimes i_{p_n})*$），那么有 $\mathrm{Tor}(f, B)$ 不依赖于 B 的投射分解的选择。

定义 3.14. If $C \in {}_RM$, apply $\mathrm{Hom}_R(\bullet, C)$ to the chosen projective resolution of B, yielding

$$\cdots \leftarrow \mathrm{Hom}(P_{n+1}, C) \xleftarrow{(d_{n+1})* \triangleq \mathrm{Hom}(d_{n+1}, C)} \mathrm{Hom}(d_n, C) \leftarrow \cdots \xleftarrow{\mathrm{Hom}(d_1, C)} \mathrm{Hom}(P_0, C) \xleftarrow{\mathrm{Hom}(d_0, C)} 0$$

with $\mathrm{Hom}(B, C)$ deleted as before. This is also a complex, the nth homology of it is called $\mathrm{Ext}^n(B, C)$.

命题 3.15. 与 Tor 函子的情况相似，Ext 函子也不依赖于投射分解的选取。

命题 3.16. If $A \in M_R, B \in {}_RM, C \in {}_RM$, then

1. $\mathrm{Tor}_0(A, B) \cong A \otimes B$
2. $\mathrm{Ext}^0(B, C) \cong \mathrm{Hom}(B, C)$
3. $\mathrm{Tor}_n(A, B) = 0 (n \geq 1)$ if A is flat or B is projective
4. $\mathrm{Ext}^n(B, C) = 0 (n \geq 1)$ if B is projective or C is injective

证明. 1. Since $A \otimes \bullet$ is right exact, then we have exact sequence

$$A \otimes P_1 \xleftarrow{i_A \otimes d_1} A \otimes P_0 \xleftarrow{i_A \otimes \pi} A \otimes B \leftarrow 0$$

Hence $\mathrm{Tor}_0(A, B) = \mathrm{Ker}(A \otimes d_0)/\mathrm{Im}(A \otimes d_1) = A \otimes P_0/\mathrm{Ker}(A \otimes \pi) \cong \mathrm{Im}(A \otimes \pi) = A \otimes B$.

2. The proof is similar to 1.

3. If A is flat, then $A \otimes P_{n+1} \to A \otimes P_n \to A \otimes P_{n-1}$ is exact since $A \otimes$ is an exact functor. Hence $\mathrm{Tor}_n(A, B) = 0$.

If B is projective, then

$$\cdots \to 0 \to \cdots \to 0 \to B \to B \to 0$$

is a projective resolution of B. Applying $A \otimes$ and deleting the $A \otimes B$ we have sequence

$$\cdots \to 0 \to \cdots \to 0 \to A \otimes B \to 0$$

Hence $\mathrm{Tor}_n(A, B) = 0$ for every $n > 0$.

4. The proof is similar to 3. □

3.2 Long Exact Sequences

定义 3.17. A chain complex will denote a complex $\mathcal{C} =< C_i, d_i >$ of Abelian groups, with $d_i : C_i \to C_{i-1}$ and with i coming in from ∞.

$$\cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots$$

定义 3.18. A cochain complex $< C_i, \partial_i >$ is a complex where $\partial : C_{i-1} \to C_i$. We can get a chain complex by replacing i with $-i$ and adjusting rhe subscript of ∂.

定义 3.19. If $\mathcal{C} =< C_i, d_i >$ and $\mathcal{C}' =< C'_i, d'_i >$ are chain complexes, then a morphism $\boldsymbol{\varphi} =< \varphi_i >$ from $\mathcal{C}$ to $\mathcal{C}'$ is a sequence of homomorphism $\varphi_i : C_i \to C'_i$ such that

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & C_i & \xrightarrow{d_i} & C_{i-1} & \longrightarrow & \cdots \\
 & & \Big\downarrow \varphi_i & & \Big\downarrow \varphi_i' & & \\
\cdots & \longrightarrow & C_i' & \xrightarrow{d_i'} & C_{i-1}' & \longrightarrow & \cdots
\end{array}
$$

commutes. A morphism of chain complexes is called a chain map. Then we get a definition of the category **Ch**. The nth homology H_n is now a additive covariant functor from **Ch** to **Ab**.

定义 3.20. A short exact sequence of chain complexes

$$0 \to \mathcal{C} \xrightarrow{\varphi} \mathcal{C}' \xrightarrow{\psi} \mathcal{C}'' \to 0$$

is a communitative diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_i & \xrightarrow{\varphi_i} & C_i' & \xrightarrow{\psi_i} & C_i'' & \longrightarrow & 0 \\
 & & \Big\downarrow d_i & & \Big\downarrow d_i' & & \Big\downarrow d_i'' & & \\
0 & \longrightarrow & C_{i-1} & \xrightarrow{\varphi_{i-1}} & C_{i-1}' & \xrightarrow{\psi_{i-1}} & C_{i-1}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

with every row is exact.

定理 3.21. Suppose

$$0 \to \mathcal{C} \xrightarrow{\varphi} \mathcal{C}' \xrightarrow{\psi} \mathcal{C}'' \to 0$$

is a short exact sequence of chain complexes. Then there is a sequence of maps $\delta_n : H_n(\mathcal{C}'') \to H_{n-1}(\mathcal{C})$ such that

$$\cdots \to H_{n+1}(\mathcal{C}'') \xrightarrow{\delta_{n+1}} H_n(\mathcal{C}) \xrightarrow{H_n(\varphi)} H_n(\mathcal{C}') \xrightarrow{H_n(\psi)} H_n(\mathcal{C}'') \xrightarrow{\delta_n} \cdots$$

is exact. The maps δ_n are called connecting homomorphisms. The sequence of maps is also natural, in that if

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C} & \xrightarrow{\varphi} & \mathcal{C}' & \xrightarrow{\psi} & \mathcal{C}'' & \longrightarrow & 0 \\
 & & \Big\downarrow \mathbf{f} & & \Big\downarrow \mathbf{g} & & \Big\downarrow \mathbf{h} & & \\
0 & \longrightarrow & \hat{\mathcal{C}} & \xrightarrow{\hat{\varphi}} & \hat{\mathcal{C}}' & \xrightarrow{\hat{\psi}} & \hat{\mathcal{C}}'' & \longrightarrow & 0
\end{array}
$$

is commutative (in **Ch**) with exact rows, then for all n,

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & H_n(\mathcal{C}'') & \xrightarrow{\delta_n} & H_{n-1}(\mathcal{C}) & \longrightarrow & \cdots \\
 & & \Big\downarrow H_n(\mathbf{h}) & & \Big\downarrow H_{n-1}(\mathbf{f}) & & \\
\cdots & \longrightarrow & H_n(\hat{\mathcal{C}}'') & \xrightarrow{\hat{\delta}_n} & H_{n-1}(\hat{\mathcal{C}}) & \longrightarrow & \cdots
\end{array}
$$

commutes.

证明. 首先给出 δ_n 的定义。

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_n & \xrightarrow{\varphi_n} & C'_n & \xrightarrow{\psi_n} & C''_n & \longrightarrow & 0 \\ & & \downarrow d_n & & \downarrow d'_n & & \downarrow d''_n & & \\ 0 & \longrightarrow & C_{n-1} & \xrightarrow{\varphi_{n-1}} & C'_{n-1} & \xrightarrow{\psi_{n-1}} & C''_{n-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

设 $x \in C''_n$,且 $x \in \operatorname{Ker}(d''_n)$。由于 ψ_n 是满射,从而存在 $y \in C'_n$ 使得 $\psi_n(y) = x$,于是 $\psi_{n-1}(d'_n(y)) = d''_n(\psi_n(y)) = d''_n(x) = 0$,即 $d'_n(y) \in \operatorname{Ker}(\psi_{n-1}) = \operatorname{Im}(\varphi_{n-1})$,由于 φ_{n-1} 是单射,从而存在唯一的 $z \in C_{n-1}$ 使得 $\varphi_{n-1}(z) = d'_n(y)$,$z = \varphi_{n-1}^{-1}(d'_n(y))$。记

$$\delta_n : H_n(\mathcal{C}'') \to H_{n-1}(\mathcal{C})$$

$$\delta_n(x + \operatorname{Im}(d''_{n+1})) = z + \operatorname{Im}(d_n)$$

接下来需要几步验证:

(1) $z+\operatorname{Im}(d_n)$ 不依赖 y 的选取。如果 $y(y') \mapsto x+\operatorname{Im}(d''_{n+1})$ 且 $\varphi_{n-1}^{-1}(d'_n(y)) = z, \varphi_{n-1}^{-1}(d'_n(y')) = z'$。

那么 $\psi_n(y - y') \in \operatorname{Im}(d''_{n+1})$,这表明存在 a 使得 $d''_{n+1}(a) = \psi_n(y - y')$。又 ψ_{n+1} 是满射,从而存在 b 使得 $d''_{n+1}(\psi_{n+1}(b)) = \psi_n(y - y')$。于是 $\psi_n(d'_{n+1}(b)) = d''_{n+1}(\psi_{n+1}(b)) = \psi_n(y - y')$,这样 $y - y' - d'_{n+1}(b) \in \operatorname{Ker}(\psi_n) = \operatorname{Im}(\varphi_n)$。即存在 c 使得 $y - y' - d'_{n+1}(b) = \varphi_n(c)$。则 $\varphi_{n-1}(d_n(c)) = d'_n(\varphi_n(c)) = d'_n(y - y' - d'_{n+1}(b)) = d'_n(y - y')$ (因为 $d'_n \circ d'_{n+1} = 0$)。于是 $d_n(c) = \varphi_{n-1}^{-1}(d'_n(y - y')) = z - z' \in \operatorname{Im}(d_n)$,这表明 $z + \operatorname{Im}(d_n) = z' + \operatorname{Im}(d_n)$。

(2) $z = \delta_n(x) \in \operatorname{Ker}(d_{n-1})$。

这是因为 $\varphi_{n-2}(d_{n-1}(z)) = d'_{n-1}(\varphi_{n-1}(z)) = d'_{n-1}(d'_n(y)) = 0$。

(3) δ_n 是一个同态。

如果 $x \in \operatorname{Im}(d''_{n+1})$,在 (1) 中已经证明如果 $y \mapsto 0 + \operatorname{Im}(d''_{n+1})$,那么 $z = \varphi_{n-1}^{-1}(d'_n(y)) \in \operatorname{Im}(d_n)$。这表明 δ_n 把零元映到零元,而 δ_n 显然是保加性的,从而 δ_n 是群同态。

接下来验证

$$\cdots \to H_{n+1}(\mathcal{C}'') \xrightarrow{\delta_{n+1}} H_n(\mathcal{C}) \xrightarrow{H_n(\boldsymbol{\varphi})} H_n(\mathcal{C}') \xrightarrow{H_n(\boldsymbol{\psi})} H_n(\mathcal{C}'') \xrightarrow{\delta_n} \cdots$$

的正合性。

(4) $\operatorname{Im}(H_n(\boldsymbol{\psi})) \subset \operatorname{Ker}(\delta_n)$

依然沿用前面的记号,不同的是,如果 $x + \operatorname{Im}(d''_{n+1}) \in \operatorname{Im}(H_n(\boldsymbol{\psi}))$,那么有 $y \in \operatorname{Ker}(d'_n)$,即 $y + \operatorname{Im}(d'_{n+1}) \in H_n(\mathcal{C}')$,此时 $H_n(\boldsymbol{\psi})(y + \operatorname{Im}(d'_{n+1})) = x + \operatorname{Im}(d''_{n+1})$。考虑到 $0 = d'_n(y) = \varphi_{n-1}(z)$,从而有 $z = 0$,即 $\delta_n(x + \operatorname{Im}(d''_{n+1})) = 0$。

(5) $\operatorname{Im}(H_n(\boldsymbol{\psi})) \supset \operatorname{Ker}(\delta_n)$

如果 $\delta_n(x+\mathrm{Im}(d''_{n+1}))=0$，即 $z\in\mathrm{Im}(d_n)$，那么存在 t 使得 $z=d_n(t)$。从而 $d'_n(y)=\varphi_{n-1}(z)=\varphi_{n-1}(d_n(t))=d'_n(\varphi_n(t))$。从而 $d'_n(y-\varphi_n(t))=0$。从而 $y-\varphi_n(t)\in\mathrm{Ker}(d'_n)$，这表明 $y-\varphi_n(t)+\mathrm{Im}(d'_{n+1})\in H_n(\mathcal{C}')$，也就是说 $\psi_n(y-\varphi_n(t)+\mathrm{Im}(d'_{n+1}))\in\mathrm{Im}(H_n(\boldsymbol{\psi}))$。而可以看出 $\psi_n(y-\varphi_n(t))=x$，从而 $x+\mathrm{Im}(d''_{n+1})\in\mathrm{Im}(H_n(\boldsymbol{\psi}))$。

(6) $\mathrm{Ker}(H_{n-1}(\boldsymbol{\varphi}))\supset\mathrm{Im}(\delta_n)$

只需证明 $\varphi_{n-1}(z+\mathrm{Im}(d_n))=\mathrm{Im}(d'_n)$，这是显然的，因为 $\varphi_{n-1}(z)=d'_n(y)\in\mathrm{Im}(d'_n)$。

(7) $\mathrm{Ker}(H_{n-1}(\boldsymbol{\varphi}))\subset\mathrm{Im}(\delta_n)$

如果 $\varphi_{n-1}(z+\mathrm{Im}(d_n))=\mathrm{Im}(d'_n)$，即 $\varphi_{n-1}(z)=d'_n(y')$。记 $\psi_n(y)=x$，如果 $x\in\mathrm{Ker}(d''_n)$，那么根据 δ_n 的构造有 $\delta_n(x)=z$。而 $x\in\mathrm{Ker}(d''_n)$ 是因为 $d''_n(x)=d''_n(\psi_n(y))=\psi_{n-1}(d'_n(y))=\psi_{n-1}(\varphi_{n-1}(z))=0$。 □

定理 3.22. Suppose $0\to A\to A'\to A''\to 0$ is exact, then for $B\in {}_RM$, there is a long exact sequence

$$\cdots\to\mathrm{Tor}_{n+1}(A'',B)\xrightarrow{\delta_{n+1}}\mathrm{Tor}_n(A,B)\to\mathrm{Tor}_n(A',B)\to\mathrm{Tor}_n(A'',B)\xrightarrow{\delta_n}\cdots\to\mathrm{Tor}_0(A'',B)\to 0$$

证明. 对 B 的投射分解 $<P_i,d_i>$ 应用3.21，由于 P_i 是投射的，由2.34是平坦的，所以

$$0\to A\otimes P_n\to A'\otimes P_n\to A''\otimes P_n\to 0$$

是正合序列。 □

定理 3.23. Suppose $0\to C\to C'\to C''\to 0$ is exact, then for $B\in {}_RM$, there is a long exact sequence

$$0\to\mathrm{Ext}^0(B,C)\to\mathrm{Ext}^0(B,C')\to\mathrm{Ext}^0(B,C'')\to\mathrm{Ext}^1(B,C)\to\cdots$$

证明. 对 B 的投射分解 $<P_i,d_i>$ 应用3.21，由于 P_i 是投射的，所以

$$0\to\mathrm{Hom}(P_n,C)\to\mathrm{Hom}(P_n,C')\to\mathrm{Hom}(P_n,C'')\to 0$$

是正合序列。 □

推论 3.24. If A' is flat, then $\mathrm{Tor}_n(A',B)=0,\forall n\geq 1$, then $0\to\mathrm{Tor}_{n+1}(A'',B)\to\mathrm{Tor}_n(A,B)\to 0$ is exact, i.e. $\mathrm{Tor}_{n+1}(A'',B)\cong\mathrm{Tor}_n(A,B)\ \forall n\geq 1$.

推论 3.25. Similarly, if C' is injective, then $\mathrm{Ext}^n(B,C'')\cong\mathrm{Ext}^{n+1}(B,C)\ \forall n\geq 1$.

推论 3.26. Suppose $B\in {}_RM$, and suppose $\mathrm{Tor}_1(R/I,B)=0$ for evert finitely generated right ideal I. Then B is flat.

证明. 考虑正合列 $0\to I\to R\to R/I\to 0$，则 $0=\mathrm{Tor}_1(R/I,B)\to\mathrm{Tor}_0(I,B)=I\otimes B\to\mathrm{Tor}_0(R,B)=R\otimes B\cong B$ 是正合的。这表明 $I\otimes B\to IB$ 是单射。根据第二章习题 11，B 是平坦的。 □

推论 3.27. Suppose $B\in {}_RM$, the following are equivalent:

- B is projective
- For all $C\in {}_RM$ and $n\geq 1$, $\mathrm{Ext}^n(B,C)=0$
- For all $C\in {}_RM$, $\mathrm{Ext}^1(B,C)=0$

证明. 3.16已经给出了 1.⇒2.，而 2.⇒3. 是显然的。

如果 3. 成立，那么类似的有 $0 \to \mathrm{Hom}(B,C) \to \mathrm{Hom}(B,C') \to \mathrm{Hom}(B,C'') \to \mathrm{Ext}^1(B,C) = 0$ 正合。这就是投射函子的定义。 □

3.3 Flat Resolution and Injective Resolution

定义 3.28. A flat resolution $< F_i, d_i >$ of $A \in {}_RM$ is an exact sequence

$$\cdots \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots F_0 \xrightarrow{\pi} A \to 0$$

where each F_n is flat. Every projective resolution is a flat resolution.

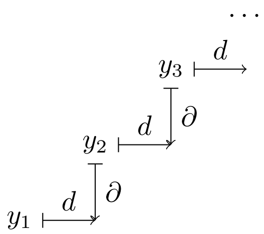
引理 3.29. Suppose $C_{ij}, d_{ij}, \partial_{ij}$ form a commutative array in **Ab** (with rows and columns being complexes)

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{3,3} & \xrightarrow{d_{3,3}} & C_{3,2} & \xrightarrow{d_{3,2}} & C_{3,1} & \xrightarrow{d_{3,1}} & C_{3,0} & \xrightarrow{d_{3,0}} & 0 \\
 & & \downarrow{\scriptstyle \partial_{3,3}} & & \downarrow{\scriptstyle \partial_{3,2}} & & \downarrow{\scriptstyle \partial_{3,1}} & & \downarrow{\scriptstyle \partial_{3,0}} & & \\
\cdots & \longrightarrow & C_{2,3} & \xrightarrow{d_{2,3}} & C_{2,2} & \xrightarrow{d_{2,2}} & C_{2,1} & \xrightarrow{d_{2,1}} & C_{2,0} & \xrightarrow{d_{2,0}} & 0 \\
 & & \downarrow{\scriptstyle \partial_{2,3}} & & \downarrow{\scriptstyle \partial_{2,2}} & & \downarrow{\scriptstyle \partial_{2,1}} & & \downarrow{\scriptstyle \partial_{2,0}} & & \\
\cdots & \longrightarrow & C_{1,3} & \xrightarrow{d_{1,3}} & C_{1,2} & \xrightarrow{d_{1,2}} & C_{1,1} & \xrightarrow{d_{1,1}} & C_{1,0} & \xrightarrow{d_{1,0}} & 0 \\
 & & \downarrow{\scriptstyle \partial_{1,3}} & & \downarrow{\scriptstyle \partial_{1,2}} & & \downarrow{\scriptstyle \partial_{1,1}} & & \downarrow{\scriptstyle \partial_{1,0}} & & \\
\cdots & \longrightarrow & C_{0,3} & \xrightarrow{d_{0,3}} & C_{0,2} & \xrightarrow{d_{0,2}} & C_{0,1} & \xrightarrow{d_{0,1}} & C_{0,0} & \xrightarrow{d_{0,0}} & 0 \\
 & & \downarrow{\scriptstyle \partial_{0,3}} & & \downarrow{\scriptstyle \partial_{0,2}} & & \downarrow{\scriptstyle \partial_{0,1}} & & \downarrow{\scriptstyle \partial_{0,0}} & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

with all rows but the bottom exact, and all columns but the rightmost exact. Then the nth homology of the bottom row is isomorphism to the nth homology of the rightmost column.

证明. 首先，定义 C 为 $C_{i,j}$ 的无交并，∂_{ij} 和 d_{ij} 都可以延拓至 C 上。若 $x \in C_{ij}$，则定义 $\partial(x) = \partial_{ij}(x)$，$d(x) = d_{ij}(x)$。

定义 $Z_n \subset \bigoplus_{i=1}^n C_{i,n-i+1}$，且 $(y_1, \cdots, y_n) \subset Z_n \iff d(y_i) = \partial(d_{i+1})$，即 $(y_1, \cdots, y_n)$ 是 $C_{1,n}$ 到 $C_{n,1}$ 中阶梯型的元素构成。



不难推出，当 $n \geq 2$ 时，如果 $(y_1, \cdots, y_n) \in Z_n$，那么 $\partial_{1,n}(y_1) \in \mathrm{Ker}(d_{0,n})$。

而如果 $x \in \mathrm{Ker}(d_{0,n})$，那么由于 $\partial_{1,n}$ 是满射，从而存在 y_1 使得 $\partial_{1,n}(y_1) = x$。设 $d_{1,n}(y_1) = z$，那么 $\partial(z) = d(x) = 0$。从而 $z \in \mathrm{Ker}(\partial_{1,n-1}) = \mathrm{Im}(\partial_{2,n-1})$，即 $\exists y_2$，$\partial(y_2) = d(y_1)$，如此往下，得到一组 $(y_1, \cdots, y_n) \in Z_n$。

这样我们得到了一个 Z_n 到 $\mathrm{Ker}(d_{0,n})$ 的满射 $(y_1, \cdots, y_n) \mapsto \partial_{1,n}(y_1)$，进而得到了 Z_n 到最底下一行的 n 阶同调群的满射 f。

现在考虑这个群的核。如果 $f(y_1, \cdots, y_n) = 0$，即 $\partial_{1,n}(y_1) \in \mathrm{Im}(d_{0,n+1})$，即存在 a 使得 $\partial_{1,n}(y_1) = d_{0,n+1}(a)$，由于 $\partial_{1,n+1}$ 是满射，从而存在 $y_1' \in C_{1,n+1}$ 使得，$\partial_{1,n+1}(y_1') = a$。此时 $\partial_{1,n}(d_{1,n+1}(y_1')) = d_{0,n+1}(a) = \partial_{1,n}(y_1)$，即 $y_1 - d_{1,n+1}(y_1') \in \mathrm{Ker}(\partial_{1,n})$。由于除了最后一列外都是正合列以及 $n \geq 2$，故 $\mathrm{Ker}(\partial_{1,n}) = \mathrm{Im}(\partial(2, n))$。即存在 $\partial_{2,n}(y_2') = y_1 - d_{1,n+1}(y_1')$。

接下来利用归纳法，如果 $y_i = \partial(s) + d(t)$，那么 $\partial(y_{i+1}) = d(y_i) = d(\partial(s) + d(t)) = d(\partial(s)) = \partial(d(s))$，那么 $y_{i+1} - d(s) \in \mathrm{Ker}(\partial_{i+1,n+i-1})$，同理由于正合列，从而 $y_{i+1} - d(s) = \partial(t')$，这样存在一列 $(y_1', \cdots, y_{n+1}')$ 使得 $d(y_i') + \partial(y_{i+1}') = y_i$。

而反过来，如果存在一列 $(y_1', \cdot, y_{n+1}')$ 使得 $d(y_i') + \partial(y_{i+1}') = y_i$，那么 $d(y_i) = d(\partial(y_{i+1}')) = \partial(d(y_{i+1'})) = \partial(y_{i+1})$，即 $(y_1, \cdots, y_n) \in Z_n$，且此时 $\partial(y_1) = \partial(d(y_1')) = d(\partial(y_1'))$，即 $f(y_1, \cdots, y_n) = 0$。

这样我们得到，f 的核与一列 $(y_1', \cdots, y_{n+1}')$ 的对应。从而最下面一行的 n 阶同调群 $H_n \cong Z_n/B_n$，其中 B_n 表示 Z_n 中可以由 $(y_1', \cdots, y_n')$ 生成的。

而反过来，最右面一列的同调群 $H_n' \cong Z_n'/B_n'$，其中 Z_n' 就是所有 Z_n 中的元素的反序 $(y_n, \cdots, y_1)$，B_n' 也是反序，这样

$$H_n \cong H_n'(n \geq 2)$$

而 $n = 0, n = 1$ 的情况很好验证。

$n = 0$ 时，$\mathrm{Im}(d_{0,1}) = \mathrm{Im}(\partial_{1,0})$ 立得。

$n = 1$ 时，记 $Z_1 = \{y_1 \in C_{1,1} : \partial_{1,0} d_{1,1}(y_1) = 0\}$，同样此时有 $y_1 = d(z_1) + \partial(z_2)$。 □

推论 3.30. 对 B 的投射分解 P_i 和 A 的平坦分解 F_i 构成的网格（依然删去 $A \otimes B$）

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F_2 \otimes P_2 & \xrightarrow{d_{3,3}} & F_2 \otimes P_1 & \xrightarrow{d_{3,2}} & F_2 \otimes P_0 & \xrightarrow{d_{3,1}} & F_2 \otimes B & \xrightarrow{d_{3,0}} & 0 \\
 & & \downarrow \partial_{3,3} & & \downarrow \partial_{3,2} & & \downarrow \partial_{3,1} & & \downarrow \partial_{3,0} & & \\
\cdots & \longrightarrow & F_1 \otimes P_2 & \xrightarrow{d_{2,3}} & F_1 \otimes P_1 & \xrightarrow{d_{2,2}} & F_1 \otimes P_0 & \xrightarrow{d_{2,1}} & F_1 \otimes B & \xrightarrow{d_{2,0}} & 0 \\
 & & \downarrow \partial_{2,3} & & \downarrow \partial_{2,2} & & \downarrow \partial_{2,1} & & \downarrow \partial_{2,0} & & \\
\cdots & \longrightarrow & F_0 \otimes P_2 & \xrightarrow{d_{1,3}} & F_0 \otimes P_1 & \xrightarrow{d_{1,2}} & F_0 \otimes P_0 & \xrightarrow{d_{1,1}} & F_0 \otimes B & \xrightarrow{d_{1,0}} & 0 \\
 & & \downarrow \partial_{1,3} & & \downarrow \partial_{1,2} & & \downarrow \partial_{1,1} & & \downarrow \partial_{1,0} & & \\
\cdots & \longrightarrow & A \otimes P_2 & \xrightarrow{d_{0,3}} & A \otimes P_1 & \xrightarrow{d_{0,2}} & A \otimes P_0 & \xrightarrow{d_{0,1}} & 0 & \xrightarrow{d_{0,0}} & 0 \\
 & & \downarrow \partial_{0,3} & & \downarrow \partial_{0,2} & & \downarrow \partial_{0,1} & & \downarrow \partial_{0,0} & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

应用引理，有 $\mathrm{Tor}_n(A,B) \cong H_n(F_k \otimes B, d_k \otimes i_B)$。

定义 3.31. Suppose $C \in {}_RM$, an injective resolution $< E_i, d_i >$ of C is an exact sequence

$$0 \to C \xrightarrow{\iota} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} E_2 \xrightarrow{d_3} \cdots$$

定理 3.32. 完全相同地有 $\mathrm{Ext}^n(B,C) = H_n(\mathrm{Hom}(B,E_n), \mathrm{Hom}(B,d_n))$。

例 7. We have $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \cong \mathbb{R}$ (as a group homomorphism), so $\mathbb{Q}$ is not projective.

证明. By using the injective resolution of $\mathbb{Z}$

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \to \cdots$$

we have

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z})/\mathrm{Im}(f : \mathrm{Hom}(\mathbb{Q},\mathbb{Q}) \to \mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z})) = \mathrm{Coker}(f)$$

$\mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z})$ is a vector space over $\mathbb{Q}$, with the same dimension of $\mathbb{R}$, a continuum. □

3.4 Consequences

命题 3.33. Suppose $0 \to B \to B' \to B'' \to 0$ is a short exact sequence in ${}_RM$, and suppose $C \in {}_RM$. Then there is a long exact sequence:

$$0 \to \mathrm{Ext}^0(B'',C) \to \mathrm{Ext}^0(B',C) \to \mathrm{Ext}^0(B,C) \xrightarrow{\delta_1} \mathrm{Ext}^1(B'',C) \to \mathrm{Ext}^1(B',C) \to \cdots$$

证明. 和前面两个长正合列的证明类似，都是应用3.21。

取 C 的内射分解 $< E_i, d_i >$，记 $0 \to \mathrm{Hom}(B, E_0) \to \mathrm{Hom}(B, E_1) \to \cdots$ 为 $\mathcal{C}$，记 $0 \to \mathrm{Hom}(B', E_0) \to \mathrm{Hom}(B', E_1) \to \cdots$ 为 $\mathcal{C}'$，记 $0 \to \mathrm{Hom}(B'', E_0) \to \mathrm{Hom}(B'', E_1) \to \cdots$ 为 $\mathcal{C}''$，则有 chain complexes 的正合列

$$0 \to \mathcal{C}'' \to \mathcal{C}' \to \mathcal{C} \to 0$$

应用3.21再由 $H_n(\mathcal{C}) \cong \mathrm{Ext}^n(B, C)$ 可得命题结论。 □

推论 3.34. If B' is projective, then $\mathrm{Ext}^n(B, C) \cong \mathrm{Ext}^{n+1}(B'', C)$.

推论 3.35. Suppose $C \in {}_RM$, the following are equivalent:

1. C is injective;
2. $\mathrm{Ext}^n(B, C) = 0$ for all $B \in {}_RM$ and $n \geq 1$;
3. $\mathrm{Ext}^1(R/I, C) = 0$ for all left ideals I;

证明. 只需证 3. ⇒ 1.。

由于 $0 \to R/I \to R \to I \to 0$ 是正合序列，从而

$$0 \to \mathrm{Hom}(R/I, C) \to \mathrm{Hom}(R, C) \to \mathrm{Hom}(I, C) \to \mathrm{Ext}^1(R/I, C) = 0$$

正合。这表明 $\mathrm{Hom}(R, C) \to \mathrm{Hom}(I, C)$ 是满射。从而对任意 $f \in \mathrm{Hom}(I, C)$，都可以提升到 $g \in \mathrm{Hom}(R, C)$，g 是 f 的原像。这样由 Bare's theorem 可得 C 是内射模。 □

定义 3.36. R^{op} is the opposite to the ring R, with the same addictive operation, but the multiplication is reversed: $a \cdot b = ba$.

命题 3.37. $\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^{R^{op}}(B, A)$

证明. 当我们按定义计算 $\mathrm{Tor}_n^{R^{op}}(B, A)$ 时，对 A 进行投射分解 $< E_n >$，由于投射模是平坦模，从而这是一个平坦分解。而我们有 $\mathrm{Tor}_n^R(A, B) \cong H_n(E_n \otimes_R B, d_n \otimes_R i_B) = H_n(B \otimes_{R^{op}} E_n, i_B \otimes_{R^{op}} d_n) = \mathrm{Tor}_n^{R^{op}}(B, A)$。 □

定理 3.38. If $0 \to B \to B' \to B'' \to 0$ is short exact, then there is a long exact sequence

$$\cdots \xrightarrow{\delta_{n+1}} \mathrm{Tor}_n^R(A, B) \to \mathrm{Tor}_n^R(A, B') \to \cdots \to \mathrm{Tor}_0^R(A, B') \to \mathrm{Tor}_0^R(A, B'') \to 0$$

推论 3.39. Suppose $A \in {}_RM$, the following are equivalent:

1. A is flat;
2. $\mathrm{Tor}_n^R(A, B) = 0$ for all $B \in M_R$;
3. $\mathrm{Tor}_1^R(A, R/I) = 0$ for every finitely generated left ideal I;

推论 3.40. Suppose $B \in {}_RM$, the following are equivalent:

1. B is flat;
2. $\mathrm{Tor}_n^R(A, B) = 0$ for all $A \in M_R$;
3. $\mathrm{Tor}_1^R(R/J, B) = 0$ for every finitely generated right ideal J;

3.5 Exercises

1. Compute $\mathrm{Tor}_n^{\mathbb{Z}_8}(\mathbb{Z}_4, \mathbb{Z}_4)$.

解. Considering the projective resolution of $\mathbb{Z}_4$:

$$\cdots \to \mathbb{Z}_8 \xrightarrow{\times 4} \mathbb{Z}_8 \xrightarrow{\times 2} \mathbb{Z}_8 \xrightarrow{\times 4} \mathbb{Z}_8 \xrightarrow{\times 2} \mathbb{Z}_4 \to 0$$

we tensor it with $\mathbb{Z}_4$ and delete the part that should be deleted, then we have:

$$\cdots \to \mathbb{Z}_4 \otimes \mathbb{Z}_8 \xrightarrow{\times 2} \mathbb{Z}_4 \otimes \mathbb{Z}_8 \xrightarrow{\times 4} \mathbb{Z}_4 \otimes \mathbb{Z}_8 \to 0$$

it's equivalent to

$$\cdots \to \mathbb{Z}_4 \to \mathbb{Z}_4 \to \mathbb{Z}_4 \to 0$$

Then $\mathrm{Tor}_0 = \mathbb{Z}_4 \otimes \mathbb{Z}_4 \cong \mathbb{Z}_4$, and $\mathrm{Tor}_{2n} = \mathbb{Z}_2, \mathrm{Tor}_{2n-1} = \mathbb{Z}_2, n \geq 1$. □

3. Suppose $< F_n, d_n >$ is a flat resolution of A. Show that the nth homology

$$\cdots \to F_2 \otimes B \to F_1 \otimes B \to F_0 \otimes B \to 0$$

is isomorphic to $\mathrm{Tor}_n(A, B)$ by the following steps:

1. Verify the case $n = 0$.
2. Verify the case $n = 1$ by the following device: Set $K = \mathrm{Im}(d_1) \subset F_0$. One has a short exact sequence $0 \to K \to F_0 \to A \to 0$, to which 3.22 applies. One also has $F_2 \to F_1 \to K \to 0$ exact, and $\otimes B$ is right exact. Play these off against each other.
3. Verify the induction step $n \to n+1$, using 3.22 again, along with the fact that $\cdots \to F_2 \to F_1 \to K \to 0$ is a flat resolution of K.

证明. 1. This is the case $n = 1$ in 3.29.

2. $F_2 \otimes B \to F_1 \otimes B \to K \otimes B \to 0$ is exact. Then $H_1 = \mathrm{Ker}(F_1 \otimes B \to F_0 \otimes B)/\mathrm{Im}(F_2 \otimes B \to F_1 \otimes B) = \mathrm{Ker}(F_1 \otimes B \to F_0 \otimes B)/\mathrm{Ker}(F_1 \otimes B \to K \otimes B)$.

Since $0 \to K \to F_0 \to A \to 0$ is exact, from 3.22 we have $\mathrm{Tor}_1(F_0, B) \to \mathrm{Tor}_1(A, B) \to \mathrm{Tor}_0(K, B) \to \mathrm{Tor}_0(F_0, B)$ is exact. F_0 is flat deduces that $\mathrm{Tor}_1(F_0, B) = 0$, hence $\mathrm{Tor}_1(A, B) \cong \mathrm{Ker}(K \otimes B \to F_0 \otimes B) = \mathrm{Ker}(F_1 \otimes B \to F_0 \otimes B)/\mathrm{Ker}(F_1 \otimes B \to K \otimes B) = H_1$.

$$\begin{array}{ccc} F_1 \otimes B & \longrightarrow & F_0 \otimes B \\ & \searrow \quad \nearrow & \\ & K \otimes B & \end{array}$$

3. For the induction step $n \to n+1$, $n \geq 1$, similarly we have $\mathrm{Tor}_{n+1}(F_0, B) = 0 \to \mathrm{Tor}_{n+1}(A, B) \to \mathrm{Tor}_n(K, B) \to 0$ is exact. Then $\mathrm{Tor}_{n+1}(A, B) \cong \mathrm{Tor}_n(K, B)$.

Considering that $\cdots \to F_2 \to F_1 \to K \to 0$ is a flat resolution of K (note that there's no F_0 here), hence we obtain $\mathrm{Tor}_n(K, B) \cong H_{n+1}(= \mathrm{Im}(F_{n+2} \otimes B \to F_{n+1} \otimes B)/\mathrm{Ker}(F_{n+1} \otimes B \to F_n \otimes B))$ by the induction hypothesis. Then $H_{n+1} \cong \mathrm{Tor}_{n+1}(A, B)$. □

7. Suppose $< E_i, d_i >$ is an injective resolution of $C \in {}_RM$, $< E'_i, d'_i >$ is an injective resolution of C', and $\varphi \in \mathrm{Hom}(C, C')$. Show that fillers φ_n exist for

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & C & \xrightarrow{\iota} & E_0 & \xrightarrow{d_1} & E_1 & \longrightarrow \cdots & \xrightarrow{d_n} & E_n & \xrightarrow{d_{n+1}} & E_{n+1} & \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_1} & & & \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_{n+1}} & \\
0 & \longrightarrow & C' & \xrightarrow{\iota'} & E'_0 & \xrightarrow{d'_1} & E'_1 & \longrightarrow \cdots & \xrightarrow{d'_n} & E'_n & \xrightarrow{d'_{n+1}} & E'_{n+1} & \longrightarrow \cdots
\end{array}$$

and that any two fillers are homotopic.

证明. Since E' is injective, then φ_0 is a filler for

$$\begin{array}{ccccc}
0 & \longrightarrow & C & \xrightarrow{\iota} & E_0 \\
 & & \downarrow{\scriptstyle\iota' \circ \varphi} & \swarrow{\scriptstyle\varphi_0} & \\
 & & E'_0 & &
\end{array}$$

If $\varphi_0 \cdots \varphi_n$ has been defined, there exists a filler φ_{n+1} for

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{Im}(d_{n+1}) \cong E_n/\mathrm{Ker}(d_{n+1}) & \hookrightarrow & E_{n+1} \\
 & & \downarrow{\scriptstyle d'_{n+1} \circ \varphi_n} & \swarrow{\scriptstyle\varphi_{n+1}} & \\
 & & E'_{n+1} & &
\end{array}$$

For any two fillers $\{\varphi_n\}$ and $\{\varphi'_n\}$, since $\varphi_0 \circ \iota = \iota' \circ \varphi = \varphi'_0 \circ \iota$, we have $\mathrm{Ker}(d_1) = \mathrm{Im}(\iota) \subset \mathrm{Ker}(\varphi'_0 - \varphi_0)$. Let Δ_0 denotes one of the fillers for

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{Im}(d_1) \cong E_0/\mathrm{Ker}(d_1) & \hookrightarrow & E_1 \\
 & & \downarrow{\scriptstyle\varphi'_0 - \varphi_0} & \swarrow{\scriptstyle\Delta_0} & \\
 & & E'_0 & &
\end{array}$$

Then $\Delta_0 \circ d_1 = \varphi'_0 - \varphi_0$.

If $\Delta_0, \cdots, \Delta_n$ has been defined, and $\varphi'_n - \varphi_n = d'_n \circ D_{n-1} + D_n \circ d_{n+1}$, then $(\varphi'_{n+1} - \varphi_{n+1} - d'_{n+1} \circ D_n) \circ d_{n+1} = d'_{n+1} \circ (\varphi' - \varphi - D_n \circ d_{n+1}) = d'_{n+1} \circ (d'_n \circ D_{n-1}) = 0$. We obtain $\mathrm{Ker}(d_{n+2}) = \mathrm{Im}(d_{n+1}) \subset \mathrm{Ker}(\varphi'_{n+1} - \varphi_{n+1} - d'_{n+1} \circ D_n)$, let Δ_{n+1} be a filler for

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{Im}(d_{n+2}) \cong E_{n+1}/\mathrm{Ker}(d_{n+2}) & \hookrightarrow & E_{n+2} \\
 & & \downarrow{\scriptstyle\varphi'_{n+1} - \varphi_{n+1}} & \swarrow{\scriptstyle\Delta_{n+1}} & \\
 & & E'_0 & &
\end{array}$$

Then φ_n and φ'_n are homotopic. □

8. Show that if $\mathrm{Ext}^1_R(B,C)=0$, then any short exact sequence $0\to C\to X\to B\to 0$ is split.

证明. Denote that $0\to C\xrightarrow{\varphi} X\xrightarrow{\psi} B\to 0$.

Using 3.23 we can obtain $0\to \mathrm{Ext}^0(B,C)\to \mathrm{Ext}^0(B,X)\to \mathrm{Ext}^0(B,B)\to \mathrm{Ext}^1(B,C)=0$ is exact. It's equivalent to $0\to \mathrm{Hom}(B,C)\to \mathrm{Hom}(B,X)\to \mathrm{Hom}(B,B)\to 0$ is exact. Since $\mathrm{Hom}(B,X)\to \mathrm{Hom}(B,B)$ is onto, there exists $f\in \mathrm{Hom}(B,X)$ such that $f\mapsto i_B$, that means, $\psi\circ f=i_B$.

Similarly using 3.33 we have exact sequence $0\to \mathrm{Hom}(B,C)\to \mathrm{Hom}(X,C)\to \mathrm{Hom}(C,C)\to 0$, and $\mathrm{Hom}(X,C)\to \mathrm{Hom}(C,C)$ deduces that $g\circ\varphi=i_C$ for some $g\in \mathrm{Hom}(X,C)$. Let $g'(x)=g(x)-g(f(\psi(x)))$, then $g'(\varphi(x))=g(\varphi(x))=i_C$ and $g'(f(x))=g(f(x))-g(f(\psi(f(x))))=0$.

The last step is to prove $\varphi\circ g'+f\circ\psi=i_X$. This is deduced easily by the exactness. Let T denote $\varphi\circ g'+f\circ\psi$, then $x-T(x)\in \mathrm{Ker}(\psi)=\mathrm{Im}(\varphi)$, that is, $x-T(x)=\varphi(y)$ for some $y\in C$. Hence $g'(x-T(x))=y$, i.e. $y=g'(f(\psi(x)))=0$. So $x=T(x)$, $T=i_X$. □

9. Suppose I is a left ideal and J is a right ideal. Show that

1. $\mathrm{Tor}_n(R/J,R/I)\cong \mathrm{Tor}_{n-2}(J,I)$ for $n>2$;
2. $\mathrm{Tor}_2(R/J,R/I)\cong \mathrm{Ker}(J\otimes I\to JI)$;
3. $\mathrm{Tor}_1(R/J,R/I)\cong (J\bigcap I)/(JI)$

证明. From exact sequence $0\to I\to R\to R/I\to 0$ and 3.38 we have $0=\mathrm{Tor}_n(R/J,R)\to \mathrm{Tor}_n(R/J,R/I)\to \mathrm{Tor}_{n-1}(R/J,I)\to \mathrm{Tor}_{n-1}(R/J,R)$ is exact.

If $n>2$, we have $\mathrm{Tor}_n(R/J,R/I)\cong \mathrm{Tor}_{n-1}(R/J,I)$. By using 3.22 and the similar process, we can obtain $\mathrm{Tor}_{n-1}(R/J,I)\cong \mathrm{Tor}_{n-2}(J,I)$.

If $n=2$, $0\to \mathrm{Tor}_1(R/J,I)\to \mathrm{Tor}_0(J,I)\to \mathrm{Tor}_0(R,J)$ is exact. Then $\mathrm{Tor}_1(R/J,I)\cong \mathrm{Im}(\mathrm{Tor}_1(R/J,I)\to \mathrm{Tor}_0(J,I))=\mathrm{Ker}(\mathrm{Tor}_0(J,I)\to \mathrm{Tor}_0(R,J))\cong \mathrm{Ker}(J\otimes I\to J)$. We have $\mathrm{Ker}(J\otimes I\to J)=\mathrm{Ker}(J\otimes I\to JI)$.

If $n=1$, $\mathrm{Tor}_1(R/J,R/I)\cong \mathrm{Im}(\mathrm{Tor}_1(R/J,R/I)\to (R/J)\otimes I)=\mathrm{Ker}(R/J\otimes I\to R/J\otimes R)\cong (I/IJ\to R/J)=(J\cap I)/JI$. □

10. Suppose B is an Abelian group. The torsion subgroup, $T(B)$, is the subgroup of B consisting of elements of finite order. Show that $T(B)\cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},B)$.

证明. We have $B/T(B)$ is torsion free, then it's flat. Considering the exact sequence $0\to T(B)\to B\to B/T(B)\to 0$ and using 3.38 we can obtain $\mathrm{Tor}_2^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},B/T(B))=0\to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},T(B))\to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},B)\to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},B/(T(B))=0$ is exact. Hence $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},T(B))\cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},B)$

From chapter 2, exercise 12 we can obtain $\mathbb{Q}$ is flat, then $0\leftarrow \mathbb{Q}/\mathbb{Z}\leftarrow \mathbb{Q}\leftarrow \mathbb{Z}\leftarrow 0\leftarrow\cdots$ is a flat resolution. By 3.30 we have $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z},T(B))=\mathrm{Ker}(\mathbb{Z}\otimes T(B)\to \mathbb{Q}\otimes T(B))\cong \mathrm{Ker}(T(B)\to \mathbb{Q}\otimes T(B))$.

$\mathbb{Q}\otimes T(B)$ is generated by $q\otimes t$, $q\in\mathbb{Q}$, $t\in T(B)$, and there exists $n\in\mathbb{Z}_+$ such that $nt=0$. This deduces $q\otimes t=q/n\otimes nt=0$, i.e. $\mathbb{Q}\otimes T(B)=0$. Then $\mathrm{Ker}(T(B)\to\mathbb{Q}\otimes T(B))=T(B)$ □

11. Show that

1. $\mathrm{Ext}^n(\oplus B_i,C)\cong\Pi\mathrm{Ext}^n(B_i,C)$
2. $\mathrm{Ext}^n(B,\Pi C_i)\cong\Pi\mathrm{Ext}^n(B,C_i)$
3. $\mathrm{Tor}_n(A,\oplus B_i)\cong\oplus\mathrm{Tor}_n(A,B_i)$

证明. 1. Apply $\mathrm{Hom}(\oplus B_i,\bullet)\cong\Pi\mathrm{Hom}(B_i,\bullet)$ to an injective resolution of C.

2. Apply $\mathrm{Hom}(\bullet,\Pi C_i)\cong\Pi\mathrm{Hom}(\bullet,C_i)$ to a projective resolution of B.

3. Apply $\bullet\otimes(\oplus B_i)\cong\oplus(\bullet\otimes B_i)$ to a flat resolution of A. □

12. Suppose B_1 and B_2 are submodules of $B\in{}_RM$. Show that $\forall C\in{}_RM$ there is a long exact sequence

$$0\to\mathrm{Hom}(B_1+B_2,C)\to\mathrm{Hom}(B_1,C)\oplus\mathrm{Hom}(B_2,C)\to\mathrm{Hom}(B_1\cap B_2,C)\to\mathrm{Ext}^1(B_1+B_2,C)\to\cdots$$

证明. This is the corollary of last exercise (since $0\to B_1\cap B_2\to B_1\otimes B_2\to B_1+B_2\to 0$ is exact). □

4 Dimension Theory

4.1 Dimension Shifting

命题 4.1. If $B\in{}_RM$, $n\geq 1$, and $\mathrm{Ext}^n(B,\bullet)\equiv 0$, then $\mathrm{Ext}^k(B,\bullet)\equiv 0$ for all $k\geq n$.

证明. For any $C\in{}_RM$, imbedding C in an injective E yields $\mathrm{Ext}^{n+1}(B,C)\cong\mathrm{Ext}^n(B,E/C)=0$. □

定义 4.2. We now define projective dimension, abbreviated $\mathrm{P}-\dim$:

$$\mathrm{P}-\dim\ B=\inf\{n\geq 0:\mathrm{Ext}^{n+1}(B,\bullet)\equiv 0\}$$

推论 4.3. If $\mathrm{P}-\dim\ B=0$, then B is projective.

证明. Use 3.27. □

命题 4.4. If $C\in{}_RM$, $n\geq 1$, and $\mathrm{Ext}^n(\bullet,C)\equiv 0$, then $\mathrm{Ext}^k(\bullet,C)\equiv 0$ for all $k\geq n$.

定义 4.5. We now define injective dimension, abbreviate $\mathrm{I}-\dim$:

$$\mathrm{I}-\dim\ C=\inf\{n\geq 0:\mathrm{Ext}^{n+1}(\bullet,C)\equiv 0\}$$

命题 4.6. If $B\in{}_RM$, $n\geq 1$, and $\mathrm{Tor}_n(\bullet,B)=0$, then $\mathrm{Tor}_k(\bullet,B)\equiv 0$ for all $k\geq n$.

定义 4.7. We define flat dimension, abbreviated $\mathrm{F}-\dim$:

$$\mathrm{F}-\dim\ B=\inf\{n\geq 0:\mathrm{Tor}_{n+1}(\bullet,B)\equiv 0\}$$

注 4.8. *For $A \in M_R$, we can define three dimensions similarly.*

定义 4.9. We now define the right global dimension of R itself, abbreviated $\mathrm{LG}-\dim$:

$$\mathrm{LG}-\dim\ R = \sup\{\mathrm{P}-\dim\ B : B \in {}_R M\}$$

Similarly, the right global dimension is defined:

$$\mathrm{RG}-\dim\ R = \sup\{\mathrm{P}-\dim\ A : A \in M_R\}$$

The weak dimension, is defined:

$$\mathrm{W}-\dim\ R = \sup\{\mathrm{F}-\dim\ B : B \in {}_R M\}$$

命题 4.10.

1. $\mathrm{LG}-\dim\ R = \inf\{n \geq 0 : \mathrm{Ext}^{n+1}(\bullet,\bullet) \equiv 0\} = \sup\{\mathrm{I}-\dim\ C : C \in {}_R M\}$
2. $\mathrm{W}-\dim\ R = \inf\{n \geq 0 : \mathrm{Tor}_{n+1}(\bullet,\bullet) \equiv 0\} = \sup\{\mathrm{F}-\dim\ A : A \in M_R\}$

命题 4.11. Suppose $0 \to D \to L_1 \to L_2 \to \cdots \to L_n \to D' \to 0$ is exact in ${}_R M$, and $d \geq 0$:

1. If $\mathrm{P}-\dim\ L_j \leq d$ for all j, then $\mathrm{Ext}^k(D,C) \cong \mathrm{Ext}^{k+n}(D',C)$ for all $C \in {}_R M$ and $k > d$.
2. If $\mathrm{I}-\dim\ L_j \leq d$ for all j, then $\mathrm{Ext}^k(B,D') \cong \mathrm{Ext}^{k+n}(B,D)$ for all $B \in {}_R M$ and $k > d$.
3. If $\mathrm{F}-\dim\ L_j \leq d$ for all j, then $\mathrm{Tor}^k(A,D) \cong \mathrm{Tor}^{k+n}(A,D')$ for all $A \in M_R$ and $k > d$.

证明. 三个命题的证明是类似的，对 n 归纳：

$n=1$ 时, $0 \to \mathrm{Ext}^k(D,C) \to \mathrm{Ext}^{k+1}(D',C) \to 0$ 是正合序列, 从而 $\mathrm{Ext}^k(D,C) \cong \mathrm{Ext}^{k+1}(D',C)$。

$n-1 \to n$: 记 Q 是 $L_n \to D'$ 的核，那么 $0 \to D \to L_1 \to \cdots \to L_{n-1} \to Q \to 0$ 和 $0 \to Q \to L_n \to D' \to 0$ 都是正合的。这样由归纳假设有 $\mathrm{Ext}^k(D,C) \cong \mathrm{Ext}^{k+n-1}(Q,C) \cong \mathrm{Ext}^{k+n}(D',C)$。 □

定义 4.12. For any projective (or flat) resolution $< P_n >$ of B, set $K_0 = B, K_1 = \mathrm{Ker}(\pi), K_n = \mathrm{Ker}(d_{n-1})$, then $\cdots \to 0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to B \to 0$ is exact. K_n is called the nth kernel of the projective resolution.

命题 4.13 (Projective Dimension Theorem)**.** Suppose $B \in {}_R M$. The following are equivalent:

1. $\mathrm{P}-\dim\ B \leq n$
2. The nth kernel of any projective resolution of B is projective
3. There exists a projective resolution of B whose nth kernel is projective.
4. There exists a projective resolution $< P_k, d_k >$ of B for which $P_k = 0$ when $k > n$

证明. 1. ⇒ 2.: $0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to B \to 0$ 正合及4.11给出 $\mathrm{Ext}^1(K_n,C) \cong \mathrm{Ext}^{n+1}(B,C)$, 由于 $\mathrm{P}-\dim\ B \leq n$，那么 $\mathrm{Ext}^1(K_n,C) \cong \mathrm{Ext}^{n+1}(B,C) = 0$，这给出 $\mathrm{P}-\dim\ K_n = 0$，即 K_n 是投射模。

2. ⇒ 3.: 这个结论是平凡的。

3. $\Rightarrow$ 4.: 既然 K_n 是投射的，那么 $\cdots \to 0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to B \to 0$ 自然是投射分解，且满足条件。

4. $\Rightarrow$ 1.: 此时 $\mathrm{Hom}(P_k, C) = 0,\ k > n$，由 $\mathrm{Ext}^k(B, C)$ 的定义可以得到 $\mathrm{Ext}^k(B, C) = 0,\ k > n$，这样 $\mathrm{P-dim}\ B \leq n$。 □

类似的，有

命题 4.14 (Flat Dimension Theorem)**.** Suppose $B \in {}_RM$. The following are equivalent:

1. $\mathrm{F-dim}\ B \leq n$
2. $\mathrm{Tor}_{n+1}(R/I, B) = 0$ for all finitely generated right ideal I
3. The nth kernel of any flat resolution of B is flat
4. There exists a flat resolution of B whose nth kernel is flat
5. There exists a flat resolution $< F_k, d_k >$ of B for which $F_k = 0$ when $k > n$

推论 4.15. For all $B \in {}_RM$, $\mathrm{F-dim}\ B \leq \mathrm{P-dim}\ B$

证明. If $\mathrm{P-dim}\ B = \infty$, it's trivial. If $\mathrm{P-dim}\ B = n$, then the nth kernel of a projective resolution of B is projective, hence flat. Thus, $\mathrm{F-dim}\ B \leq n$. □

推论 4.16. $\mathrm{LG-dim}\ R \geq \mathrm{W-dim}\ R,\ \mathrm{RG-dim}\ R \geq \mathrm{W-dim}\ R$

定义 4.17. Similarly for injectives, suppose we are given an injective resolution of $C \in {}_RM$:

$$0 \to C \xrightarrow{\iota} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} E_2 \to \cdots$$

Set $D_n = \mathrm{Im}(d_n) = E_{n-1}/\mathrm{Ker}(d_n) = E_{n-1}/\mathrm{Im}(d_{n-1}),\ n \geq 1$, $D_0 = \mathrm{Im}(\iota) \cong C$, then $0 \to C \to E_0 \to \cdots \to E_{n-1} \to D_n \to 0 \to \cdots$ is exact. D_n is called the nth cokernel of the injective resolution.

命题 4.18 (Injective Dimension Theorem)**.** Suppose $C \in {}_RM$. The following are equivalent:

1. $\mathrm{I-dim}\ C \leq n$
2. $\mathrm{Ext}^{n+1}(R/I, C) = 0$ for all left ideals I
3. The nth cokernel of any injective resolution of C is injective
4. There exists an injective resolution of C whose nth cokernel is injective.
5. There exists an injective resolution $< E_k, d_k >$ of C for which $E_k = 0$ when $k > n$

命题 4.19 (Global dimension Theorem)**.** $\mathrm{LG-dim}\ R = \sup\{\mathrm{P-dim}\ R/I : I\ a\ left\ ideal\}$

推论 4.20. If $\mathrm{LG-dim}\ R > 0$, then $\mathrm{LG-dim}\ R = 1 + \sup\{\mathrm{P-dim}\ I : I\ a\ left\ ideal\}$.

证明. We have $\mathrm{Ext}^n(I, C) \cong \mathrm{Ext}^{n+1}(R/I, C),\ n \geq 1$. □

推论 4.21. $\mathrm{LG-dim}\ R \leq 1 \iff$ every left ideal is projective.

推论 4.22. If R is PID, then $\mathrm{LG-dim}\ R \leq 1$.

注 4.23. *The ring* $\mathbb{Z}_4$ *is a principal ideal ring but not a domain, and* $\mathrm{W}-\dim R = \infty$ *since* $\mathrm{Tor}_n(\mathbb{Z}_2, \mathbb{Z}_2) \neq 0$.

命题 4.24 (Weak dimension Theorem)**.**

$\mathrm{W}-\dim R = \sup\{\mathrm{F}-\dim R/I : I \textit{ a finitely generated right ideal}\}$

$\mathrm{W}-\dim R = \sup\{\mathrm{F}-\dim R/I : I \textit{ a finitely generated left ideal}\}$

推论 4.25. If $\mathrm{W}-\dim R > 0$, then

$\mathrm{W}-\dim R = 1 + \sup\{\mathrm{F}-\dim I : I \textit{ a finitely generated right ideal}\}$

$\mathrm{W}-\dim R = 1 + \sup\{\mathrm{F}-\dim I : I \textit{ a finitely generated left ideal}\}$

推论 4.26. $\mathrm{W}-\dim R \leq 1 \iff$ every finitely generated left ideal is flat.

4.2 When Flats are Projective

命题 4.27 (Projective Basis Theorem)**.** Suppose $P \in {}_RM$. The following are equivalent:

1. P is projective
2. If P is generated by $\{s_i : i \in I\}$, then there exists $\varphi_i \in P^* = \mathrm{Hom}(P, R), i \in I$ such that for all $x \in P$, $\{i \in I : \varphi_i(x) \neq 0\}$ is finite, and $x = \sum \varphi_i(x)s_i$.
3. There exists a generating set $\{s_i : i \in I\}$ of P for which there exist $\varphi_i \in P^*, i \in I$ such that for all $x \in P$, $\{i \in I : \varphi_i(x) \neq 0\}$ is finite, and $x = \sum \varphi_i(x)s_i$.

证明. 1. $\Rightarrow$ 2.: Suppose P is generated by $\{s_i : i \in I\}$, let $F = \otimes_{i\in I}R$ be the free module on I, $\pi : F \to P$ defined via $i \mapsto s_i$. Then $F \to P \to 0$ is exact, hence $0 \to \ker(\pi) \to F \to P \to 0$ splits since P is projective (id_P could extend to $\eta : P \to F$). Suppose the ith coordinate of $\eta(x)$ is φ_i. Then $x = \sum \varphi_i(x)s_i$ and $\{i \in I : \varphi_i(x) \neq 0\}$ is finite.

2. $\Rightarrow$ 3.: It is trivial.

3. $\Rightarrow$ 1.: Let F is the free module on I, and $\pi : i \to s_i$. define $\eta(x) = \sum \varphi_i s_i$, then $i_P = \pi\eta$, then $F \to P \to 0$ is splits. Thus, P is a direct summand of F, hence projective. □

推论 4.28. Suppose P is finitely generated. Then P is projective if and only if the image of the natural map $P^* \otimes P \to \mathrm{Hom}(P, P)$ contains i_P.

证明. $\sum \varphi_i \otimes s_i \mapsto i_P \iff x = \sum \varphi_i(x)s_i$. □

定义 4.29. Suppose $B \in {}_RM$ is finitely generated. B is called finitely presented provided there exists a finitely generated free module F, and a map π from F onto B, such that $\mathrm{Ker}(\pi)$ is also finitely generated.

推论 4.30. All finitely generated projective modules are finitely presented.

证明. Projective module $B \in {}_RM$ gives the sequence $0 \to \mathrm{Ker}(\pi) \to F \to B \to 0$ splits, then $\mathrm{Ker}(\pi)$ is a direct summand, hence a image. Therefore $\mathrm{Ker}(\pi)$ is finitely generated. □

命题 4.31. Suppose $B \in {}_R M$ is flat, and suppose $C \in {}_R M$ is finitely presented. Then $C^* \otimes B \to \mathrm{Hom}(C, B)$ is an isomorphism.

证明. We may suppose that we have finitely generated free modules F_0 and F_1, and an exact sequence $F_1 \to F_0 \to C \to 0$ ($\mathrm{Ker}(F_0 \to C)$ is the image of F_1). Since $\mathrm{Hom}(\bullet, R)$ is left exact, then $0 \to C^* \to F_0^* \to F_1^*$ is exact. Then $0 \to C^* \otimes B \to F_0^* \otimes B \to F_1^* \otimes B$ is exact since B is flat.

If F is finitely generated free module, i.e. $F = \oplus_{i=1}^n R$, then $F^* \otimes B \cong \oplus(R \otimes B) \cong \oplus B \cong \oplus \mathrm{Hom}(R, B) = \mathrm{Hom}(F, B)$. This lemma gives $F_1^* \otimes B \cong \mathrm{Hom}(F_1, B), F_0^* \otimes B \cong \mathrm{Hom}(F_0, B)$. Then there exists commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & C^* \otimes B & \longrightarrow & F_0^* \otimes B & \longrightarrow & F_1^* \otimes B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}(C, B) & \longrightarrow & \mathrm{Hom}(F_0, B) & \longrightarrow & \mathrm{Hom}(F_1, B)
\end{array}$$

We can obtain the conclusion by 5-lemma. □

定理 4.32. Suppose $P \in {}_R M$ is finitely generated. The following are equivalent:

1. P is projective
2. P is flat and finitely presented
3. The natural map from $P^* \otimes P$ to $\mathrm{Hom}(P, P)$ is an isomorphism
4. The image of the natural map from $P^* \otimes P$ to $\mathrm{Hom}(P, P)$ contains i_P

证明. 4.30 gives 1. ⇒ 2., 4.31 gives 2. ⇒ 3., 3. ⇒ 4. is trivial, 4.28 gives 4. ⇒ 1.. □

推论 4.33. Suppose R is left Noetherian, and suppose B is a finitely generated left R-module. Then $\mathrm{P} - \dim\ B = \mathrm{F} - \dim\ B$.

证明. Choose F_0 a finitely generated free module and $\pi : F_0 \to B$ is onto. Since R is left Noetherian, hence $\mathrm{Ker}(\pi)$ is also finitely generated. Then choose F_1 a finitely generated free module like previous, etc.

We get a series of finitely generated free modules $\{F_n\}$ such that $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to B \to 0$ is a flat resolution of B. Then the nth kernel of this resolution is finitely generated, hence obviously finitely presented, hence flat if and only if projective. Then $\mathrm{P} - \dim\ B = \mathrm{F} - \dim\ B$. □

推论 4.34. Suppose R is left Noetherian. Then $\mathrm{LG} - \dim\ R = \mathrm{W} - \dim\ R$.

证明. The two dimensions only rely on the quotients R/I, which are finitely generated modules. □

推论 4.35 (Auslander)**.** Suppose R is both right and left Noetherian. Then $\mathrm{LG} - \dim\ R = \mathrm{RG} - \dim\ R$.

4.3 Dimension Zero

定义 4.36. A Dedekind domain is an integral domain with global dimension less than or equal to one.

推论 4.37. Every ideal of Dedekind domain is projective.

证明. 4.21. □

命题 4.38. Suppose R is commutative, and I is a projective ideal containing a nonzero divisor b. Then I is finitely generated, say by $s_1, \cdots, s_n$. Further, there exists $b_1, \cdots, b_n$ in R such that, for all j, b divides xb_j for all $x \in I$, and $x = \sum(xb_j/b)s_j$. In particular, if R is an integral domain, then any projective ideal is finitely generated; hence, any Dedekind domain is Noetherian.

证明. Since I is projective, suppose I is generated by s_i, then there exists $\varphi_i : I \to R$ such that $x = \sum \varphi_i(x)s_i$ for all $x \in I$. If $\varphi_i(b) = 0$, then $x\varphi_i(b) = b\varphi_i(x) = 0$. Since b is nonzero divisor, then $\varphi_i(x) = 0$, that is, there exists finite $\varphi_i \neq 0$, i.e. I is finitely generated.

Let $\varphi_i(b) = b_i \neq 0$, then b divides xb_i, and $\varphi_i(x) = xb_i/b$ (the quotient is well-defined if b is the denominator). □

定义 4.39. If R is a ring, and B is an R-module (left or right), then B is semisimple if every submodule of B is a direct summand of B.

命题 4.40. Since 4.10, if $\mathrm{LG}-\dim R = 0$, then every left R-module is injective. Then if B is a submodule of C, we have B is a direct summand of C (2.43). Then every left R-module is semisimple.

定义 4.41. If R is ring, and B is an R-module, then B is simple if $B \neq 0$, and the only submodules of B are 0 and B.

B is simple if and only if $B \neq 0$ and $Rx = B$ for all $x \in B$.

定义 4.42. If R is a ring, and B is an R-module, B' is a submodule. Then B' is maximal if B/B' is simple.

命题 4.43. Since $Rx = R/\mathrm{Ann}(x)$, then B is simple if and only if B is isomorphism to a quotient R/I, where I is a maximal left ideal.

引理 4.44. Every submodule of a semisimple module is semisimple.

证明. 设 D 是半单的，且 C 是 D 的子模。对 C 的任意子模 B，存在 A 使得 $A \oplus B = D$。这样 $A \cap B = \varnothing$，且 $A + (B \cap C) = C$。于是 $C = A \oplus (B \cap C)$。 □

命题 4.45. Suppose R is a ring, and $B \in {}_RM$. Suppose B is generated by a set S together with an element x, but is not generated by S alone. Then any submodule of B that contains S, and is maximal with respect of the property of not containing x, is maximal as a submodule. Such submodules exist.

证明. Use Zorn's lamma. □

推论 4.46. Every nonzero semisimple module contains a simple submodule.

证明. For $x \neq 0$, let B' be the submodule generated by x, and let $S = \varnothing$. Then there exists B'' is a maximal submodule of B'. Since B' is semisimple, then $B' = B'' \oplus B'''$, i.e. $B''' \cong B'/B''$ is simple. □

命题 4.47. Every semisimple module is the sum of its simple submodule.

证明. If B is semisimple, let B' denote the sum of all the simple submodules of B. If $B \neq B'$, then $B = B' \oplus B''$, then B'' is semisimple, which contains a simple submodule. Then $B'' \cap B' \neq 0$, contradiction. We obtain $B' = B$. □

引理 4.48. Suppose B is an R-module, I is an index set, and B_i is a simple submodule of B for each $i \in I$. Also suppose $B = \sum_I B_i$, that is, B is the sum of the B_i (probably not direct) of the B_i. Then for any submodules of B there exists a subset J of I such that $B = B' \oplus (\oplus_{i\in J} H_i)$.

证明. Consider the set Σ consists of all the subsets J of I such that $B' + (\sum_{i\in J} B_i) = B' \oplus (\oplus_{i\in J} B_i)$, $\Sigma \neq \varnothing$ since $\varnothing \in \Sigma$. Use Zorn's lemma, there exists a maximal element (also said J) in Σ.

Suppose $t \in I - J$, then $B' \oplus (\oplus_{\in J\cup\{t\}} B_i) \neq B' + (\sum_{i\in J\cup\{t\}} B_i) = B' \oplus (\oplus_{i\in J} B_i) + B_t$, that is, $B_t \cap (B' \oplus (\oplus_{i\in J} B_i)) \neq 0$. As a nonempty submodule of simple module B_t, we have $B_t \cap (B' \oplus (\oplus_{i\in J} B_i)) = B_t$. Then for all B_t, $B_t \in B' \oplus (\oplus_{i\in J} B_i)$, hence $B = \sum B_t \subset B' \oplus (\oplus_{i\in J} B_i)$, i.e. $B = B' \oplus (\oplus_{i\in J} B_i)$. □

定理 4.49. Suppose B is an R-module. The following are equivalent:

1. B is semisimple
2. B is a sum of simple submodules
3. B is a direct sum of simple submodules

证明. This theorem follows from 4.47 and 4.48. □

定义 4.50. For all R-module B, $\mathrm{Hom}(B, B)$ is a ring, called the endomorphism ring of B, and is denoted $\mathrm{End}(B)$.

命题 4.51. If B and B' are simple R-modules, then every nonzero element of $\mathrm{Hom}(B, B')$ is an isomorphism.

证明. For every $0 \neq \varphi \in \mathrm{Hom}(B, B')$, B and B' are simple gives that $\mathrm{Ker}(\varphi) = 0$ and $\mathrm{Im}(\varphi) = B'$. □

推论 4.52 (Schur's Lemma)**.** If B is a simple R-module, then $\mathrm{End}(B)$ is a division ring.

注 4.53. 除环是指所有非零元都可逆的环，交换除环即为域。

命题 4.54. 显然 $\mathrm{Hom}(B^n, B^n)$ 中的元素可以看做 $\mathrm{Hom}(B, B)$ 中的元素构成的 $n\times n$ 的矩阵 $M_n(\mathrm{End}(B))$，且矩阵运算与同态的运算保持一致。这样，我们有 $\mathrm{End}(B^n) \cong M_n(\mathrm{End}(B))$

推论 4.55. Suppose $B_1, \cdots, B_N$ are pairwise nonisomorphism simple R-modules. Then

$$\mathrm{End}(B_1^{n_1} \oplus \cdots B_N^{n_N}) \cong M_{n_1}(\mathrm{End}(B_1)) \oplus \cdots M_{n_N}(\mathrm{End}(B_N))$$

引理 4.56. Suppose B is a finitely generated semisimple R-module. Then B is a finite direct sum of simple modules.

证明. B is a direct sum of simple submodules $B = \oplus_{i\in I} B_i$

If B is generated by $x_1, \cdots, x_n$, then for every $1 \le j \le n$, the number of i satisfied $x_j \in B_i$ is finite. Then there exists a finite subset $J \subset I$ such that $B = \oplus_{i\in J} B_i$. □

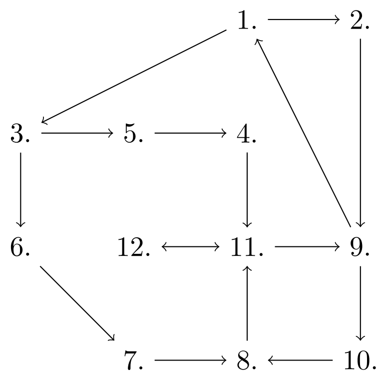
引理 4.57. If R is a ring, then the opposite ring to $M_n(R)$ is isomorphic to $M_n(R^{op})$, via $A \mapsto A^T$.

引理 4.58. $\mathrm{Hom}_R(R, R) \cong R^{op}$ (as a ring isomorphism)

定理 4.59 (Artin-Wedderburn Structure Theorem)**.** Suppose R is a ring. The following are equivalent:

1. $\mathrm{LG} - \dim R = 0$
2. Every left R-module is projectrive.
3. Every left R-module is injective.
4. Every left R-module is semisimple.
5. Every short exact sequence of left R-modules splits.
6. Every left ideal is injective.
7. Every maximal left ideal is injective.
8. Every maximal left ideal is a direct summand of R.
9. For every left ideal I, R/I is projective.
10. Every simple left R-module is projective.
11. R is semisimple as a left R-module.
12. R is finite direct sum of matrix rings over division rings.

证明. 思维导图如下:



我们来一条一条说明：

1. ⇒ 2. 是因为 LG − dim R 的定义。

1. ⇒ 3. 由 LG − dim R 的进一步性质4.10给出。

2. ⇒ 9. 由4.19给出。

3. ⇒ 5. 由2.43给出。

3. ⇒ 6.：左理想自然是左模，从而是内射模。

4. ⇒ 11.：R 自然是一个左 R-模，从而半单。

5. ⇒ 4.：若 B 是一个左 R-模且 A 是 B 的子模，考虑正合列 $0 \to A \to B \to B/A \to 0$。因为这是分裂序列，从而 $B = A \oplus B/A$，这样 B 是半单的。

6. ⇒ 7.：这是平凡的。

7. ⇒ 8. 由2.43给出。

8. ⇒ 11.：设 J 是所有 R 的单子模的和，如果 $J \neq R$，那么存在一个极大理想 I 包含 J，但是 R/I 是单的，且 $R/I \cap J = \varnothing$，矛盾。从而 $J = R$，由4.49，R 是半单的。

9. ⇒ 1. 由4.19给出。

9. ⇒ 10. 由4.43给出。

10. ⇒ 8.：对极大左理想 I，R/I 是单左模，它是投射模。由2.31得出。

11. ⇒ 9.：$R = R/I \oplus I$，这样 R/I 是自由模的直和项，显然是投射的。

11. ⇒ 12.：R 显然是有限生成的 R-模，这样由4.56R 是单子模的有限直和，这样 $R^{op} \cong \mathrm{Hom}_R(R, R)$ 是矩阵环的有限直和。两边同时取 op 即得。

12. ⇒ 11.：设 $R = M_{r_1}(R_1) \oplus \cdots \oplus M_{r_n}(R_n)$。对每个 $M_r(R)$，记 L_k 为除了第 k 列外均为 0 的矩阵构成的理想，这样 L_k 互不相交且单且直和为 $M_r(R)$，这样 $M_r(R)$ 是半单的，这样 R 是半单的。 □

推论 4.60. 可以看出，12. 和左模右模无关，从而 LG − dim $R = 0 \iff$ RG − dim $R = 0$。

定义 4.61. If R is a ring, then R is regular if, for all $a \in R$, there exists $r \in R$ for which $a = ara$ (r depends on a).

推论 4.62. 显然 $Rra \subset Ra$，而 $Ra = Rara \subset Rra$，这样 $Ra = Rra$。而 $ra = (ra)^2$，这样每个主理想都是幂等元生成的。

引理 4.63. Suppose R is a ring, and I is a left ideal. Then I is a direct summand of R if and only if I is principal and generated by an idempotent.

证明. If $I = Re$ with $e = e^2$, let $f = 1 - e$, then $Re + Rf = R$. And we have $Re \cap Rf = 0$ since $r_1 e = r_2(1-e) \Rightarrow (r_1 + r_2)e = r_2 \Rightarrow (r_1 + r_2)e = (r_1 + r_2)e^2 = r_2 e \Rightarrow r_1 e = 0$. Then $R = (Re) \oplus (Rf)$.

If $R = I \oplus J$, then $1 = e + f$ for some $e \in I$, $f \in J$, and $ef \in I \cap J \Rightarrow ef = 0$. Therefore $e^2 + f^2 = e + f$, hence $e - e^2 = f^2 - f \in I \cap J$, i.e. $e = e^2$, $f = f^2$. $Re \subset I$, $Rf \subset J$, and $Re + Rf = R$ deduces that $Re = I$, $Rf = J$. □

引理 4.64. Suppose R is a ring, and suppose e and f are idempotents in R such that $ef = 0 = fe$. Then $e + f$ is idempotent and $Re + Rf = R(e + f)$.

引理 4.65. $Ra + Rb = Ra + Rb(1 - a)$

引理 4.66. Suppose R is regular. Then every finitely generated left ideal is principal (and generated by an idempotent).

证明. 事实上4.64和4.65给出两个主理想的和还是主理想

这是因为 $Ra+Rb = Ra+Rb(1-a)$，记 $b' = rb(1-a)$，其中 r 是使得 $b(1-a)rb(1-a) = b(1-a)$ 成立的 r。这样 $Rb' = Rb(1-a)$。且 $b'a = 0$, $b'^2 = rb(1-a)\cdot rb(1-a) = rb(1-a) = b'$。设 $a' = a(1-b')$，则 $a'^2 = a(1-b')a(1-b') = a(a-b'a)(1-b') = a^2(1-b') = a(1-b')$，且 $b'a' = 0, a'b' = 0$。从而 $Ra + Rb = Ra + Rb' = Ra' + Rb' = R(a' + b')$。 □

定理 4.67 (Weak Dimension Zero Characterization)**.** Suppose R is a ring. The following conditions are equivalent:

1. $\mathrm{W}-\dim\, R = 0$
2. Every left R-module is flat
3. For every finitely generated left ideal I, R/I is projective
4. $\mathrm{Tor}_1(R/J, R/I) = 0$ for every finitely generated right ideal J and every finitely generated left ideal I
5. $\mathrm{Tor}_1(R/aR, R/Ra) = 0$ for every $a \in R$
6. R is regular

证明. 1. ⇒ 2. 由 $\mathrm{W}-\dim\, R$ 的定义给出。

2. ⇒ 4. 这是平凡的。

4. ⇒ 5. 这是平凡的。

5. ⇒ 6. 根据第三章的题 9，$0 = \mathrm{Tor}_1(R/J, R/I) \cong (J \cap I)/(JI)$，这样 $J \cap I = JI$。从而 $a \in aR \cap Ra = aRRa = aRa$，即存在 r 使得 $ara = a$。

6. ⇒ 3.: R 是 regular 给出 R 的所有有限生成理想都是主理想，且由幂等元生成。这样4.63给出 I 是 R 的直和项，$R = I \oplus R/I$，即 R/I 是投射的。

3. ⇒ 1.: 3.39给出任意的左 R-模都是平坦的，这样 $\mathrm{W}-\dim\, R = 0$。 □

注 4.68. *Since R/I is finitely presented (I is finitely generated), then R/I is projective if and only if it's flat.*

4.4 An Example

定义 4.69. A Bézout domain is an integral domain in which every finitely generated ideal is principal.

命题 4.70. Since $R \to Ra$ is a module isomorphism, then every principal ideal in integral domain is projective, hence flat.

推论 4.71. Any finitely generated ideal of a Bézout domain is projective, hence flat.

推论 4.72. 4.26给出 $\mathrm{W}-\dim R \leq 1$。

命题 4.73. If a Bézout domain is not a PID, then there exists an ideal I which is not finitely generated. By 4.38 we have I is not projective. Then 4.21 deduces $\mathrm{LG}-\dim R \geq 2$. Then Bézout domain is an example satisfied $\mathrm{W}-\dim \neq \mathrm{LG}-\dim$.

命题 4.74. If I is a nonprincipal ideal of a Bézout domain, and I is generated by a countable set $\{r_i\}$, then $\mathrm{P}-\dim I = 1$.

证明. Let $I_n = (r_1, \cdots, r_n) \triangleq Ra_n$, $I_{n+1} \supset I_n$ deduces that $a_{n+1} | a_n$. We have $I = \bigcup I_n$. Suppose $a_n = d_n a_{n+1}$.

Denote $F = \oplus_{i=1}^{\infty} R$, and send $(x_1, \cdots, x_n, \cdots)$ to $\sum x_j a_j$. The map is onto.

Set $v_1 = (1, -d_n, 0, \cdots), v_2 = (0, 1, -d_2, 0, \cdots), \cdots$, we have $v_n \mapsto 0$, so $v_n \in \mathrm{Ker}(F \to I) \triangleq K$.

Suppose $(x_1, \cdots, x_N, 0, \cdots) \mapsto 0$, then $\sum_{i=1}^{N} x_i a_i = 0$. Since it's a finite sum, by induction on N, we have K is generated by v_n.

And if $\sum_{i=1}^{M} v_i s_i = 0$, then by induction on M, we have $s_i \equiv 0$, then K is free. Then $\cdots \to 0 \to K \to F \to I \to 0$ is a projective resolution of K, hence $\mathrm{P}-\dim I \leq 1$.

But I is not projective, hence $\mathrm{P}-\dim I = 1$. □

4.5 Exercises

2. Suppose $0 \to B \to B' \to B'' \to 0$ is short exact in ${}_R M$, and suppose $\mathrm{P}-\dim B > \mathrm{P}-\dim B'$ or $\mathrm{P}-\dim B'' > 1 + \mathrm{P}-\dim B'$. Show that $\mathrm{P}-\dim B'' = 1 + \mathrm{P}-\dim B$.

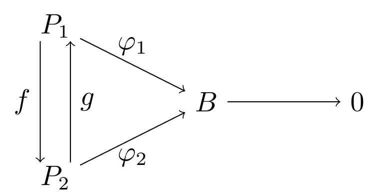
证明. When $\mathrm{P}-\dim B' = \infty$, the conclusion is trivial.

Let $\mathrm{P}-\dim B' = t < \infty$.

For every $C \in_R M$, $\cdots \to \mathrm{Ext}^n(B'', C) \to \mathrm{Ext}^n(B', C) \to \mathrm{Ext}^n(B, C) \to \mathrm{Ext}^{n+1}(B'', C) \to \cdots$ is exact. Then whenever $\mathrm{P}-\dim B > \mathrm{P}-\dim B'$ or $\mathrm{P}-\dim B'' > \mathrm{P}-\dim B' + 1$, we have $0 \to \mathrm{Ext}^t(B, C) \cong \mathrm{Ext}^{t+1}(B'', C)$. □

4. Prove Schanuel's lemma: If $0 \to K_i \to P_i \to B \to 0$ are short exact for $i = 1, 2$, with P_1 and P_2 projective, then $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

证明. Since P_1 and P_2 are projective, there exists f and g such that



We have $K_1 \cong \mathrm{Ker}(\varphi_1)$, $K_2 \cong \mathrm{Ker}(\varphi_2)$, and $x \in \mathrm{Ker}(\varphi_1) \Rightarrow x \in \mathrm{Ker}(\varphi_2 \circ f) \Rightarrow f(x) \in \mathrm{Ker}(\varphi_2)$, similarly $g(\mathrm{Ker}(\varphi_2)) \in \mathrm{Ker}(\varphi_1)$. Then we can build a homomorphism

$$\psi : \mathrm{Ker}(\varphi_1) \oplus P_2 \to P_1$$

$$(k_i, p_i') \mapsto k_i - g(p_i')$$

If $k_i - g(p_i') = 0$, then $k_i = g(p_i')$, then $\varphi_2(p_i') = 0$, so we have $\mathrm{Ker}(\psi) \cong K_2$. Furthermore, for all $p_i \in P_2$, $k_i \triangleq p_i - g(f(p_i)) \in \mathrm{Ker}(\varphi_1)$, this deduces ψ is onto. Since P_1 is projective, then P_1 is a summand of $\mathrm{Ker}(\varphi_1) \oplus P_2$, i,e. $P_1 \oplus K_2 \cong K_1 \oplus P_2$. □

5. Suppose B is finitely presented, and suppose P is projective and finitely generated, with $0 \to K \to P \to B \to 0$ short exact. Show that K is finitely generated.

证明. There are two short exact sequences $0 \to K \to P \to B \to 0$ and $0 \to \mathrm{Ker}(\oplus R \to B) \to \oplus R \to B \to 0$. By the last exercise, we have $K \oplus (\oplus R) \cong \mathrm{Ker}(\oplus R \to B) \oplus P$. Hence $K \cong (P \oplus \mathrm{Ker}(\oplus R \to B))/(\oplus R)$ is finitely generated. □

6. A ring R is called a Boolean ring if $x = x^2$ for all $x \in R$.

1. Show that any Boolean ring R is commutative, with $x = -x$ for all $x \in R$
2. Show that any Boolean ring is regular
3. Show that any finite Boolean ring is isomorphic to a finite direct sum of copies of $\mathbb{Z}_2$

证明. 1. Since $x + x = (x + x)^2 = x^2 + x^2 + x^2 + x^2 = x + x + x + x \Rightarrow x + x = 0 \Rightarrow x = -x$, we have $x + y = (x + y)(x + y) = x + y + xy + yx \Rightarrow xy + yx = 0 \Rightarrow xy = yx$ for all $x, y \in R$.

2. Let $r = 1$, we have $a = ara$.

3. For $x \in R$, let $y = 1 - x$. Then $Ax + Ay = A$, and $Ax \cap Ay = 0$. Then $Ax \oplus Ay = A$. By induction we can get the conclusion. □

7. Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$, the product of infinitely many copies of $\mathbb{Z}_2$. Note that R is a Boolean ring, hence is regular by exercise 6. Let $I = \bigoplus_{i=1}^{\infty} \mathbb{Z}_2$.

1. Show that I is projective but not finitely generated.
2. Show that R/I is flat and finitely generated, but neither finitely presented nor projective. Show that as explicitly as possible where 4.32 break down.
3. Show that $\mathrm{LG}-\dim R > \mathrm{W}-\dim R$.

证明. 1. This is equivalent to $\mathbb{Z}_2$ is projective. But $R = \mathbb{Z}_2 \oplus (\prod \mathbb{Z}_2)$ is project, then $\mathbb{Z}_2$ is projective.

2. 4.67 deduces R/I is flat. R is finitely generated as a R-module, then R/I is finitely generated. Since I is not finitely generated, R/I is not finitely oresented.

3. $\mathrm{W}-\dim R = 0$. Since R/I is not projective, this deduces $\mathrm{LG}-\dim R > 0$. □

8. Suppose I is a projective ideal in a GCD domain, and suppose $x, y \in I$. Show that "the" GCD d of x and y belongs to I. Hence show that I is principal. Finally, deduce that a GCD domain is a Dedekind domain if and only if it is a PID.

证明. We need some new theories to prove I is finitely generated.

定义 4.75. R 是一个整环，K 是它的分式域。R 的**分式理想**是 K 的一个 R-子模 M 满足 $xM \subseteq R$，对某个 $x \in R, x \neq 0$ 成立。特别通常意义上的理想（现称整理想）是分式理想。若 M 是一分式理想，所有满足 $xM \subseteq R$ 的 $x \in K$ 的集合记为 $(R : M)$。
K 的一个 R-子模 M 叫做**可逆理想** (invertible ideal)，如果存在 K 的一个 R-子模 N 使得 $MN = R$。N 此时是唯一的且 $N = (R : M)$。

命题 4.76. 可逆理想是有限生成的。

证明. 设 $1 = \sum_{i=1}^n m_i n_i$，这样 $\forall x \in M, x = 1 \cdot x = (\sum_{i=1}^n m_i n_i)x = \sum (n_i x) m_i$ 是有限生成的。 □

命题 4.77. 分式理想 M 可逆等价于投射。

证明. 如果 M 是可逆的，那么 $1 = \sum m_i n_i$，且有 m_i 生成了 M。定义 $f_i : M \to R$ $f_i(m) = n_i m$，则 $m = \sum f_i(m) m_i$。由4.27，可得 M 是投射的。
如果 M 是投射的，那么存在 m_i 和 $f_i \in \mathrm{Hom}(M, R)$ 使得 $m = \sum f_i(m) m_i$。给定一个 $b \in M$，令 $k_i = f_i(b)/b$。对任意 $m \in M$，令 $m = p/q$, $b = r/s$, $p, q, r, s \in R$，那么 $mf_i(b)sq = pf_i(r) = f_i(pr) = rf_i(p) = sbf_i(qm) = sbqf_i(m)$，那么 $mf_i(b) = bf_i(m)$，这样 $k_i m = f_i(m)$。从而 $k_i M \subset R$。设 $f_1(b), \cdots, f_n(b)$ 不为 0，此时有 $m = \sum f_i(m) m_i = k_i m m_i$，这表明 $1 = \sum k_i m_i$，从而 M 可逆。 □

推论 4.78. R 的投射理想是可逆的。

Let $I = (a_1, \cdots, a_n)$, let d be the $gcd(a_1, \cdots, a_n)$. And we have I is invertible, that is, there exists J such that $J = (R : I)$ is a submodule of K. For all $x/y \in K$, $(x, y) = 1$, $x/y \in (R : I) \Leftrightarrow xa_i/y \in R, \forall i \Leftrightarrow y | a_i x, \forall i \Leftrightarrow y | gcd(xa_1, \cdots, xa_n) = xd$.

Then we need a lemma of GCD domain: If $gcd(a, b) = 1$ and $a|bc$, then $a|c$. The proof is easy: since $gcd(ac, bc) = c$, we have $a|c$.

By the lemma, we have $y|xd \Leftrightarrow y|d$, then $J = y^{-1}R$. Then $1 = 1/d(\sum x_i a_i)$, that is $d = \sum x_i a_i \in I$. Then $I = Rd$ is principal.

Then GCD domain is a Dedekind domain if and only if every ideal is projective, this is equivalent to every ideal is principal ideal. □

9. The following is a theorem from commutative algebra:

Suppose R is a UFD, and not a field. Then R is a PID if and only if the Krull dimension of R is equal to one.

Prove the analogous result with the word "Krull" replaced by "weak".

证明. Firstly we prove the theorem in commutative algebra.

If R is PID, then for any prime ideal's chain $0 \subsetneq (p) \subsetneq (q)$, we have $q|p$, hence $p = q$, contradiction. Then the Krull dimension of R is equal to one.
If the Krull dimension of R is equal to one, then every nonzero prime ideal is maximal, then every prime ideal is principal.
Then we can get the result by the following lemma:
A ring is a principal ideal ring if and only if every prime ideal is principal.

证明. Let S be the ideals which are not principal, assume $S \neq \varnothing$. By Zorn's lemma, there's a maximal element I in S. If I is not prime, then there exists $ab \in I$ and $a \notin I$, $b \notin I$. Hence $(a)+I, (b)+I \notin S$, that is, $(a)+I = (x)$, $(b)+I = (y)$. Then $(xy) = (ab)+((a)+(b))I+I = I$, contradiction. □

Then for the "weak" one:

If R is UFD and $\mathrm{W}-\dim R = 1$, then 4.25 deduces $\mathrm{F}-\dim I = 0$ for all finitely generated ideal I. Hence it's flat. UFD is GCD domain, hence finitely generated ideal is principal. □

10.Prove the module law: If A, B and C are submodules of D, with $A \subset C$, then $A + (B \cap C) = (A + B) \cap C$.

证明. It's trivial. □

11. Suppose $B_i \in {}_RM$. Show that $\mathrm{P}-\dim(\oplus B_i) = \sup(\mathrm{P}-\dim B_i)$.

证明. This is deduces by chapter 3, exercise 11. □

12. Suppose R is an integral domain and suppose a and b are nonzero and are nonunits in R. Set $\bar{R} = R/Rab$, and if $x \in R$, set $\bar{x} \in \bar{R}$.

1. Show that $\bar{R}\bar{b} \cong \bar{R}/\bar{R}\bar{a}$
2. Show that the following are equivlant:
    - $\bar{R}/\bar{R}\bar{a}$ is $\bar{R}$-projective
    - $Ra + Rb = R$
    - $Ra + Rb = R$ and $Ra \cap Rb = Rab$
    - $\bar{R}\bar{a} \oplus \bar{R}\bar{b} = \bar{R}$
3. Show that if $Ra + Rb \neq R$, then $\bar{R}$ has infinite weak dimension.
4. Compute $\mathrm{Tor}_n^{\bar{R}}(\bar{R}/\bar{R}\bar{a}, \bar{R}/\bar{R}\bar{a})$ for the case $R = \mathbb{Z}[x]$, $a = x$, $b = 2$

证明. 1. For all $x \in R$, let f maps $\bar{x}$ to $\bar{x} + \bar{b}\bar{R}$. If $\bar{x} \in \bar{b}\bar{R}$, then $x \in aR$, hence $\mathrm{Ker}(f) = \bar{a}\bar{R}$.

2. (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) is trivial, and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) is also obvious enough.

3. $\bar{R}/\bar{R}\bar{a} \cong \bar{R}\bar{b}$ is not $\bar{R}$-projective, then $\mathrm{Tor}_1(\bar{R}/\bar{R}\bar{a}, \bar{R}/\bar{R}\bar{a}) \neq 0$, similarly $\mathrm{Tor}_1(\bar{R}/\bar{R}\bar{b}, \bar{R}/\bar{R}\bar{b}) \neq 0$, hence $\mathrm{Tor}_{2n+1}(\bar{R}/\bar{R}\bar{a}, \bar{R}/\bar{R}\bar{a}) \neq 0$ by the chapter 3, exercise 9. Then $\bar{R}$ has infinite weak dimension.

4. Use the chapter 3, exercise 9. □

13. Suppose P is projective and finitely generated in ${}_RM$, and suppose $C \in {}_RM$. Show that $P^* \otimes C \to \mathrm{Hom}(P, C)$ is an isomorphism.

证明. We first prove that P^* is projective. Let R^n is the free finitely generated module contains P, then P is the direct summand. Then $\mathrm{Hom}(P, R)$ is the direct summand of $\mathrm{Hom}(R^n, R) \cong R^n$, hence projective.

P^* and P are finitely generated projective. Then for all free resolution of C: $\cdots \to F_1 \to F_0 \to C \to 0$, we can replace F_i with $P^* \otimes F_i$ and $\mathrm{Hom}(P, F_i)$. Since for free module F we have $P^* \otimes F \cong \mathrm{Hom}(P, F)$, then by 5-lemma the result is true. □

14. Suppose $\mathrm{P}-\dim B = N \geq n$. Show that the nth kernel of any projective resolution of B has projective dimension $N - n$.

证明. For any projective resolution of B

$$\cdots \to P_k \to P_{k-1} \to \cdots \to P_0 \to B \to 0$$

this induces a projective resolution of $\mathrm{Im}(P_k \to P_{k-1})$

$$\cdots \to P_k \to \mathrm{Im}(P_k \to P_{k-1}) \to 0$$

Then by 4.13, $\mathrm{P}-\dim \mathrm{Im}(P_k \to P_{k-1}) = N - k$ □

15. Analytical similar objects can be algebraically quite different.

1. Let $R = C^\infty(\mathbb{R})$. Let M be the maximal ideal $\{f \in R : f(0) = 0\}$. Show that $\mathrm{P}-\dim R/M = 1$.
2. Let $R = C(\mathbb{R})$. Let M be the maximal ideal $\{f \in R : f(0) = 0\}$. Show that $\mathrm{P}-\dim R/M > 1$.

证明. 1. We have projective resolution

$$\cdots \to 0 \to R \xrightarrow{\times x} R \xrightarrow{f \mapsto f(0)} R/M \to 0$$

then $P_k = 0$ when $k > 1$, then we can obtain $\mathrm{P}-\dim R/M = 1$ by 4.13.

The resolution is exact since $f(0) = 0 \Rightarrow f = x(f'(0) + \frac{1}{2}f''(0)x + \cdots)$.

2. If $\mathrm{P}-\dim R/M \leq 1$, then $\mathrm{P}-\dim M = 0$, hence M is projective, hence for all I is an ideal of R we have $0 = \mathrm{Tor}_1(R/I, M) = \mathrm{Ker}(I \otimes M \to IM) = 0$, i.e. $I \otimes M \cong IM$. Therefore $M \otimes M \cong M^2$. But obviusly there exists $f, g \in M$ such that $f \neq 0,\ g \neq 0$ but $f \otimes g \mapsto fg = 0$. □

5 Change of Rings

5.1 Computational Considerations

定义 5.1. A covariant functor $F : {}_SM \to {}_RM$ is called "strong additivity", if $F(\oplus B_i) \cong \oplus F(B_i)$.

命题 5.2. Suppose $F : {}_SM \to {}_RM$ is an exact, strongly additive covariant functor. Then for all $B \in {}_SM$:

1. $\mathrm{P}-\dim\ _RF(B) \le \mathrm{P}-\dim\ _SB + \mathrm{P}-\dim\ _RF(S)$

2. $\mathrm{F}-\dim\ _RF(B) \le \mathrm{P}-\dim\ _SB + \mathrm{F}-\dim\ _RF(S)$

证明. If B is free, then $B = \oplus S$ and $\mathrm{P}-\dim\ _SB = 0$, hence $F(B) = \oplus F(S)$. Since chapter 4 erercise 11 we have $\mathrm{P}-\dim\ _R \oplus F(S) = \mathrm{P}-\dim\ _RF(S)$, or zero if the index set is empty.

If B is projective, then $\mathrm{P}-\dim\ _SB = 0$, and $B \oplus C = S$ is free. Then we have $\mathrm{P}-\dim\ _RF(B) \le \mathrm{P}-\dim\ _RF(B) \oplus F(C) = \mathrm{P}-\dim\ _RF(B \oplus C) \le \mathrm{P}-\dim\ _RF(S)$ by the first case.

If $\mathrm{P}-\dim\ _SB = \infty$, it's trivial. If $\mathrm{P}-\dim\ _SB = n$, let $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to B$ is the projective resolution, then $0 \to F(P_n) \to \cdots \to F(P_0) \to F(B) \to 0$ is exact. Then by 4.11 and the second case, we have $\mathrm{P}-\dim\ _RF(B) = \mathrm{P}-\dim\ _RF(P_n) + n \le \mathrm{P}-\dim\ _RF(S) + n$. □

定义 5.3. If $B \in {}_RM$, and B' is a submodule, define the "Supremal Projective Dimension" of (B', B) as follows:

$$\mathrm{SP}-\dim\ (B', B) = \sup\{\mathrm{P}-\dim\ C : C \ is\ a\ submodule\ of\ B,\ and\ C \supset B'\}$$

Set $\mathrm{SP}-\dim\ B = \mathrm{SP}-\dim\ (0, B)$.

命题 5.4. If $\mathrm{LG}-\dim\ R > 0$, then $\mathrm{LG}-\dim\ R = 1 + \mathrm{SP}-\dim\ R$.

命题 5.5. Suppose $B \in {}_RM$, B' is a submodule of B, and B'' is a submodule is a submodule of B''. Then

$$\mathrm{SP}-\dim\ (B'', B) = \max\{\mathrm{SP}-\dim\ (B'', B'), \mathrm{SP}-\dim\ (B', B)\}$$

证明. Obviously we have $\mathrm{SP}-\dim\ (B'', B) \ge \max\{\mathrm{SP}-\dim\ (B'', B'), \mathrm{SP}-\dim\ (B', B)\}$.

If the inequality is strict, then there exists C satisfies

$$\mathrm{P}-\dim\ C > \max\{\mathrm{SP}-\dim\ (B'', B'), \mathrm{SP}-\dim\ (B', B)\} \ge \max\{\mathrm{P}-\dim\ C \cap B', \mathrm{P}-\dim\ C + B'\}$$

But this cannot happen, choose n such that $\mathrm{P}-\dim\ C \ge n > \max\{\mathrm{P}-\dim\ C \cap B', \mathrm{P}-\dim\ C + B'\}$ and make D satisfy $\mathrm{Ext}^n(C, D) \ne 0$, hence $\mathrm{Ext}^n(C + B', D) = \mathrm{Ext}^n(C \cap B', D) = 0$. But chapter 3, exercise 12 gives an exact sequence $0 = \mathrm{Ext}^n(C + B', D) \to \mathrm{Ext}^n(C, D) \oplus \mathrm{Ext}^n(B', D) \to \mathrm{Ext}^n(C \cap B', D) = 0$, a contradiction. □

推论 5.6. If $\mathrm{LG}-\dim\ R > 0$, and $0 = I_0 \subset I_1 \subset \cdots \subset I_n = R$ is a chain of left ideals in R, then $\mathrm{LG}-\dim\ R = 1 + \max\{\mathrm{SP}-\dim\ (I_{j-1}m, I_j)\}$.

命题 5.7. Suppose $B, C \in {}_RM$, then

$$\mathrm{SP}-\dim\ (B \oplus C) = \max\{\mathrm{SP}-\dim\ B, \mathrm{SP}-\dim\ C\}$$

证明. We have $\mathrm{SP}-\dim\ (B \oplus C) = \max\{\mathrm{SP}-\dim\ (B \oplus 0), \mathrm{SP}-\dim\ (B \oplus 0, B \oplus C)\}$. But any submodule between $B \oplus 0$ and $B \oplus C$ coresponds to a submodule of C, so any module between $B \oplus 0$ and $B \oplus C$ has the form $B \oplus C'$. Since $\mathrm{P}-\dim\ (B \oplus C') = \max\{\mathrm{P}-\dim\ B, \mathrm{P}-\dim\ C\}$, then $\mathrm{SP}-\dim\ (B \oplus 0, B \oplus C) = \max\{\mathrm{P}-\dim\ B, \mathrm{SP}-\dim\ C\}$. Hence $\mathrm{SP}-\dim\ (B \oplus C) = \max\{\mathrm{SP}-\dim\ B, \mathrm{SP}-\dim\ C\}$. □

推论 5.8. If $\mathrm{LG}-\dim R > 0$, and if $R = I_1 \oplus \cdots \oplus I_n$ is a direct sum of left ideals, then $\mathrm{LG}-\dim R = 1 + \mathrm{SP}-\dim R = 1 + \max\{\mathrm{SP}-\dim I_j\}$.

命题 5.9. Suppose $\phi : R \to \hat{R}$ is a surjective ring homomorphism, and suppose $\hat{R}$ is R-projective. Then $\mathrm{P}-\dim {}_R\hat{B} = \mathrm{P}-\dim {}_{\hat{R}}\hat{B}$ for all $\hat{B} \in {}_{\hat{R}}M$.

证明. We can infer $\mathrm{P}-\dim {}_R\hat{B} \leq \mathrm{P}-\dim {}_{\hat{R}}\hat{B}$ from 5.2, hence all $\hat{R}$-projective modules are R-projective.

Suppose $\hat{B}$ is R-projective. Suppose $\hat{P}$ is a $\hat{R}$-projective and there exists a surjective $\hat{\pi} : \hat{P} \to \hat{B}$. As two projective R-modules, $\hat{B}$ is the direct summand of $\hat{P}$. Thus there exists an R-module homomorphism $\hat{\eta} : \hat{B} \to \hat{P}$ satisfying $\hat{\pi}\hat{\eta} = i_{\hat{B}}$. ϕ is surjective gives $\hat{\eta}$ is an $\hat{R}$-module homomorphism, hence $\hat{B}$ is a direct summand as $\hat{R}$-modules. Hence $\hat{B}$ is R-projective $\iff$ $\hat{B}$ is $\hat{R}$-projrctive.

In general, it is easily observed by 4.13 and the previous conclusion. □