Abstract

The Mordell conjecture was proposed in 1922 by Louis Mordell, and was firstly proved in 1983 by Gerd Faltings. Mordell conjectured that for any smooth projective curve C/K of genus ≥ 2 with K a number field, the set $C(K)$ of K-rational points is finite.

In 2020, Brian Lawrence and Akshay Venkatesh present a new proof [LV20] by their p-adic period mapping, borrowing some portion of Faltings's original proof. The main goal of this thesis is to effectively present how they worked on this beautiful result.

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Chapter 1

An overview of the proof

In 1922, Mordell proposed a conjecture about the number of rational points on a curve.

Conjecture 1.0.1 (Mordell)**.** *Given a smooth and projective curve C over a number field K with genus ≥ 2, the number of its rational points $\#C(K)$ is finite.*

Remark 1.0.2. *These following results [Sta23, Tag 0B45] and [Sta23, Tag 0A26] tell us that the conditions "projectiveness" and "properness" for a smooth curve Y/K coincide.*

Remark 1.0.3. *By a variety over K we mean a geometrically integral, separated scheme of finite type over K.*

By a curve we mean a variety with dimension 1.

In 1968, Parshin ([Par68]) made the first breakthrough that the Mordell conjecture is a consequence of the Shafarevich conjecture.

Conjecture 1.0.4 (The Shafarevich conjecture for curves)**.** *There exist only finitely many (smooth, projective) curves (up to isomorphism) defined over K of genus g and with good reduction outside of S, where S is a finite set of places.*

Conjecture 1.0.5 (The Shafarevich conjecture for Abelian varieties)**.** *There exists only finitely many Abelian varieties over K (up to isomorphism) of dimension g with good reduction outside S, and with a polarization of degree d.*

Remark 1.0.6. *Note that by taking the Jacobians and Torelli theorem ([LS01] Appendix), 1.0.5 implies 1.0.4.*

In 1966, Tate proved the following result for Abelian varieties over a finite field k and their Tate modules.

Theorem 1.0.7 (Tate conjecture for finite fields, [Tat66])**.** Let k be a finite field, and ℓ a prime number with $\ell \neq \text{char}\ (k)$, then

- for any two Abelian varieties A, B over k, the natural map

$$\mathbb{Z}_\ell \otimes \text{Hom}(A, B) \to \text{Hom}(T_\ell A, T_\ell B)^{\text{Gal}(k_s/k)}$$

  is an isomorphism. Here the right hand is the group of $\mathbb{Z}_\ell$-linear maps fixed by $\text{Gal}(k_s/k)$.

- for any Abelian variety A over k, the representation induced by the Tate module

$$\rho_\ell : \mathrm{Gal}(k_s/k) \to \mathrm{GL}(V_\ell A)$$

is semisimple, here k^s denotes the separable closure of k.

Indeed, he showed that these statements hold true for an arbitrary field k if we can prove the following finiteness condition:

Up to isomorphism, there are only finitely many Abelian varieties B over k such that there is an isogeny $A \to B$ of degree a power of ℓ.

In 1986, Faltings ([Fal86]) constructed a height defined on the moduli space of Abelian varieties. Through this height, he proved a slightly weaker finiteness condition that is strong enough to conclude the Tate conjecture for number fields K. Moreover, using this height, he proved that the Tate conjecture for K implies the Shafarevich conjecture. Then, the Mordell conjecture was finally proved.

The method of Faltings-Parshin can be roughly summarized as the following:

$$\begin{array}{c}
\{\text{finiteness of } p\text{-divisible groups as representations}\}(\text{Faltings finiteness theorem}) \\
\Big\downarrow \text{Tate conjecture} \\
\{\text{finiteness of Abelian varieties under isogeny}\} \\
\Big\downarrow \text{Faltings height and Raynaud theory} \\
\{\text{finiteness of Abelian varieties under isomorphism}\} \\
\Big\downarrow \text{Taking Jacobians} \\
\{\text{finiteness of curves}\} \\
\Big\downarrow \text{Taking Parshin family} \\
\{\text{finiteness of rational points}\}
\end{array}$$

In 2020, a new proof [LV20] arose. The basic idea behind this proof is totally different with Faltings, and it is more interesting that it could be generalized to varieties with higher dimensions. We will introduce this method in the following chapters of this thesis. This method can be roughly summarized as the following:

$$\begin{array}{c}
\{\text{finiteness of } p\text{-adic étale cohomology of fibers}\}(\text{Faltings finiteness theorem} + p\text{-adic Hodge theory}) \\
\Big\downarrow \text{Taking a special family} \\
\{\text{finiteness of rational points}\}
\end{array}$$

1.1 The Main Theorem

In this section, we will provide a detailed description of the main idea behind the proof from Lawrence and Venkatesh by admitting some facts. We will conclude a general result for an arbitrary smooth variety Y, and then we hope to apply it to a smooth projective curve C with genus ≥ 2.

We may use slightly different notations with [LV20].

1.1.1 The original idea

1. Basic Concepts and Basic Properties. We first introduce some fundamental results.

Fix a number field K.

Definition 1.1.1. Let S be a finite set of places of K. The ring of S-integers is the subring of K defined as $\mathcal{O}_{K,S} = \{x \in K : \forall v \notin S, v(x) \geq 0\}$.

Let Y be a smooth variety over K. All of our work is based on some families over Y; thus, we make an assumption first.

> *(A) Assume that we are given a specific smooth proper K-morphism $f : X \to Y$, such that f extends to a smooth proper morphism $\mathcal{X} \to \mathcal{Y}$ between smooth models over $\mathcal{O} \triangleq \mathcal{O}_{K,S}$, where S is a fixed finite set of places containing all Archimedean values of K and all ramified places of $K/\mathbb{Q}$.*

Remark 1.1.2. *Under the condition (A), we can associate each $y \in Y(K)$ with some p-adic Galois representations ρ_y determined by the action of G_K on the étale cohomology groups $H^*_{\text{ét}}(X_y \times_K \overline{K}, \mathbb{Q}_p)$, where p is a prime such that no place in S is above p.*

Instead of analyzing $C(K)$ for some curves C, we will first analyze $Y(K)$. In fact, we will control the size of a smaller set $\mathcal{Y}(\mathcal{O})$ (see 1.1.3).

Lemma 1.1.3 (Rational points and integral points)**.** *The set of rational points $Y(K)$ is strongly linked to geometric points of $\mathcal{Y}$. Here we list some results for rational and integral points of Y and $\mathcal{Y}$.*

Fix a place v of K.

(1) The universal property of fiber product tells us that there are bijections

$$\mathcal{Y}(K_v) \to Y(K_v)$$

$$\mathcal{Y}(K) \to Y(K)$$

(2) Since $\mathcal{O}_v \triangleq \mathcal{O}_{K_v}$ is a discrete valuation ring, by the valuation criterion of separateness, there exists an inclusion

$$\mathcal{Y}(\mathcal{O}_v) \to \mathcal{Y}(K_v)$$

If further we assume that the morphism $\mathcal{Y} \to \operatorname{Spec}(\mathcal{O})$ is proper, the inclusion $\mathcal{Y}(\mathcal{O}_v) \to \mathcal{Y}(K_v)$ is then a bijection.

(3) ([Poo23] Theorem 3.2.13) There is an inclusion

$$\mathcal{Y}(\mathcal{O}) \to \mathcal{Y}(K)$$

If further $\mathcal{Y} \to \operatorname{Spec}(\mathcal{O})$ is proper, the inclusion is a bijection.

Corollary 1.1.4. If Y is proper over K, the natural map $\mathcal{Y}(\mathcal{O}) \to Y(K)$ is indeed bijective. Then any element $y \in Y(K)$ can extend to an element in $\mathcal{Y}(\mathcal{O})$, which is also denoted by y.

Now we assume that two points $y, y_0 \in Y(K)$ can extend to a point $y, y_0 \in \mathcal{Y}(\mathcal{O})$, our main interest lies in this case. (Here, we treat y_0 as a fixed point, and y is a variable.) This condition implies that the fiber X_y admits a smooth model $\mathcal{X}_y$ over $\mathcal{O}$. We will find a way to bound $\mathcal{Y}(\mathcal{O})$. The idea is to prove the finiteness of representations ρ_y and the fiber-finiteness of the functor $y \mapsto \rho_y$. Fortunately, we have an important finiteness theorem from Faltings 1.1.5.

Lemma 1.1.5 (Faltings)**.** *Fix S and $d \geq 0$. Let K be a number field, with a finite set of places S. For each $v \notin S$, let Z_v be a finite set of elements in $\mathbb{Q}_p$. Then there are finitely many isomorphism classes of Galois representations $\rho : G_K \to \mathrm{GL}_d(\mathbb{Q}_p)$ such that*

(1) ρ is unramified outside S.

(2) For $v \notin S$, the trace of the Frobenius element Frob_v lies in Z_v.

(3) ρ is semi-simple.

Proof. For a detailed proof, one can see [Lan13] Theorem 4.3. ■

The representations $H^*_{\text{ét}}(X_y \times_K \overline{K}, \mathbb{Q}_p)$ satisfy the conditions (1) in 1.1.5 in the following sense.

Theorem 1.1.6 ([Del72] Théorème XII.5.1)**.** If X_y has good reduction outside S, then $H^*_{\text{ét}}(X_y \times_K \overline{K}, \mathbb{Q}_p)$ is unramified outside $S \cup \{p\}$.

The representations $H^*_{\text{ét}}(X_y \times_K \overline{K}, \mathbb{Q}_p)$ satisfy the conditions (2) in 1.1.5 in the following sense.

Theorem 1.1.7 ([Del80] Weil conjecture)**.** For a smooth proper variety X over a finite field $\mathbb{F}_q$, the Frobenius element Frob_q acts on $H^*_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)$. Then the characteristic polynomial P_{Frob_q} has integer coefficients and each root of it is a q-Weil number of weight w.

Under condition (A), the only remaining thing we need to deal with is that ρ_y may be non-semi-simple; thus we make the second assumption

> *(B) For the family $X \to Y$ we constructed in (A), ρ_y is semi-simple for almost all $y \in \mathcal{Y}(\mathcal{O})$.*

2. Some Reductions. To bound $\mathcal{Y}(\mathcal{O})$, we need to split it into several parts. Fix one place v of K. For any two points $y, y_0 \in \mathcal{Y}(\mathcal{O})$, we say $y \equiv y_0 (\mathrm{mod}\ v)$ if y and y_0 have the same image under the map $\mathcal{Y}(\mathcal{O}) \to \mathcal{Y}(\mathbb{F}_v)$. By the universal property of fiber product, there is a bijection $\mathcal{Y}(\mathbb{F}_v) \to \mathcal{Y}_{\mathbb{F}_v}(\mathbb{F}_v)$. Since $\mathcal{Y}_{\mathbb{F}_v}$ is of finite type over $\mathrm{Spec}(\mathbb{F}_v)$, it is covered by finitely many affine spectra of $\mathbb{F}_v$-finitely generated algebras, then $\mathcal{Y}_{\mathbb{F}_v}(\mathbb{F}_v)$ is finite. Therefore, $\mathcal{Y}(\mathcal{O})$ can be written as a finite union of residue classes.

$$\begin{tikzcd}
\mathrm{Spec}(\mathbb{F}_v) \arrow[r] \arrow[drr] & \mathrm{Spec}(\mathcal{O}) \arrow[r, shift left, "y"] \arrow[r, shift right, "y_0"'] & \mathcal{Y} \arrow[r] & \mathrm{Spec}(\mathcal{O}) \\
 & & Y_{\mathbb{F}_v} \arrow[u] \arrow[r] & \mathrm{Spec}(\mathbb{F}_v) \arrow[u]
\end{tikzcd}$$

Define the period residue disk to be one of these classes $\Omega_v = \Omega_{v,y_0} \triangleq \{y \in \mathcal{Y}(\mathcal{O}) | y \equiv y_0 (\mathrm{mod}\ v)\}$. In addition, we will use a larger domain $\Omega'_v = \{y \in \mathcal{Y}(\mathcal{O}_v) : y \equiv y_0 (\mathrm{mod}\ v)\}$ in the K_v-analytic variety $Y(K_v)$.

$$\begin{tikzcd}
 & & & & K_v\text{-analytic variety} \arrow[d, dotted] \\
 & \Omega'_v \arrow[r, hook] & \mathcal{Y}_v(\mathcal{O}_v) \arrow[rr, hook] & & Y_{K_v}(K_v) \\
\Omega_v \arrow[ur, hook] \arrow[r, hook] & \mathcal{Y}(\mathcal{O}) \arrow[ur, hook] \arrow[r, hook] & Y(K) \arrow[urr, hook] & &
\end{tikzcd}$$

Under conditions (A) and (B), we can represent $\mathcal{Y}(\mathcal{O})$, or Ω_v, as a finite union of the fibers of the functor $y \mapsto \rho_y$ by 1.1.5. Then, to control the size of the whole set, we reduce to control the size of each fiber $\{y \in \Omega_v : \rho_y \cong \rho_{y_0}\}$.

However, the global Galois representation ρ_y is difficult to analyze. We need to make further reductions, like, we're more willing to discuss the induced local Galois representations. By restricting ρ_y to a "nice" place v , we can obtain a representation

$$\rho_{y,v} : G_{K_v} \hookrightarrow G_K \to \mathrm{Aut}\left(H^*_{\text{ét}}((X_y)_{\overline{K}}, \mathbb{Q}_p)\right)$$

Thus, it suffices to show that the functor

$$y \in \Omega_v \mapsto \rho_y \mapsto \rho_{y,v}$$

has finite fibers.

Remark 1.1.8. *If we assume that $v \nmid p$, the result will be boring because we lose abundant information about ρ_y. Indeed, by smooth-proper base change theorem, the unramified representations $H^*_{\text{ét}}$ of different fibers in some "étale neighborhood" are isomorphic. Thus, we may require $v|p$. Then every cohomology group $H^*_{\text{ét}}((X_y)_{\overline{K}}, \mathbb{Q}_p)$ defines a p-adic Galois representation of G_{K_v}. To deal with these representations, we will use the p-adic Hodge theory discovered by Faltings and Fontaine.*

Theorem 1.1.9 ([Fal88]). Let X be a smooth K_v-scheme that admits a proper model $\mathcal{X}/\mathcal{O}_v$. Then $H^*_{\text{ét}}(X_{\overline{K_v}}, \mathbb{Q}_p) \in \mathbf{Rep}^{\text{cris}}_{\mathbb{Q}_p}(G_{K_v})$ and its image under the (fully faithful) functor $D_{\text{cris}} : \mathbf{Rep}^{\text{cris}}_{\mathbb{Q}_p}(G_{K_v}) \to \mathbf{MF}^{\varphi}_{K_v}$ is

$$(H^*_{\text{cris}}(X_{\mathbb{F}_v}/W(\mathbb{F}_v))[1/p] : H^*_{\text{dR}}(X/K_v), \varphi, \mathrm{Fil}^{\bullet}_{\text{Hodge}})$$

Here $\mathbf{Rep}^{\text{cris}}_{\mathbb{Q}_p}(G_{K_v})$ denotes the category of all crystalline $\mathbb{Q}_p$-representations of G_{K_v} and $\mathbf{MF}^{\varphi}_{K_v}$ the category of all filtered φ-modules over K_v. We often write it as a simplified version

$$(H^*_{\text{dR}}(X/K_v), \varphi, \mathrm{Fil}^{\bullet}_{\text{Hodge}})$$

In summary, we have proved the following result.

Proposition 1.1.10. Let $(H^*_{\text{dR}}(X_y/K_v), \varphi_y, \mathrm{Fil}^{\bullet}_y)$ denote the image of ρ_y under the functor D_{cris}. Let

$$\Sigma_v = \Sigma_{v,y_0} = \{y \in \Omega_v : (H^*_{\text{dR}}(X_y/K_v), \varphi_y, \mathrm{Fil}^{\bullet}_y) \cong (H^*_{\text{dR}}(X_{y_0}/K_v), \varphi_{y_0}, \mathrm{Fil}^{\bullet}_{y_0})\}$$

then $\mathcal{Y}(\mathcal{O})$ is a finite union of subsets with this type. In particular, if Σ_{y_0} is finite for all $y_0 \in \mathcal{Y}(\mathcal{O})$, then $\mathcal{Y}(\mathcal{O})$ is finite.

3. Parallel Transportation and Period Mapping. We analyze the elements in Σ_{y_0} in this part. In the category $\mathbf{MF}^{\varphi}_{K_v}$, $(V_1, \varphi_1, F^1) \cong (V_2, \varphi_2, F^2)$ if and only if there exists a K_v-linear map $f : V_1 \to V_2$ such that $f \circ \varphi_1 = \varphi_2 \circ f$ and $f(F^1) = F^2$. Thus, for $y, y_0 \in \mathcal{Y}(\mathcal{O})$, we obtain the following result.

Proposition 1.1.11. The images of y and y_0 under the functor in 1.1.10 are isomorphic if and only if there exists a K_v-linear isomorphism $f : H^*_{\mathrm{dR}}(X_y/K_v) \to H^*_{\mathrm{dR}}(X_{y_0}/K_v)$ respecting the Frobenius endomorphism and the respective filtrations.

The advantage of taking Ω_v rather than $\mathcal{Y}(\mathcal{O})$ is that we have an "abstract" isomorphism

$$\nabla^*_{y,y_0} : H^*_{\mathrm{dR}}(X_y/K_v) \cong H^*_{\mathrm{dR}}(X_{y_0}/K_v)$$

between $H^*_{\mathrm{dR}}(X_y/K_v)$ and $H^*_{\mathrm{dR}}(X_{y_0}/K_v)$ as K_v-vector spaces for $y, y_0 \in \Omega'_v = \{y \in \mathcal{Y}_v(\mathcal{O}_v) : y \equiv y_0 (\mathrm{mod}\ v)\}$. This is called the p-adic parallel transportation, which will be discussed in chapter 2.

Thus, we can identify every $(H^*_{\mathrm{dR}}(X_y/K_v), \varphi_y, \mathrm{Fil}^\bullet_y)$ as an object with the underlying vector space $H^*_{\mathrm{dR}}(X_{y_0}/K_v)$. About this isomorphism, an important fact is 2.3.3, which tells us that this identity sends φ_y to φ_{y_0}, i.e., this isomorphism respects the Frobenius. Therefore,

$$\begin{aligned}
&(H^*_{\mathrm{dR}}(X_y/K_v), \varphi_y, \mathrm{Fil}^\bullet_y) \cong (H^*_{\mathrm{dR}}(X_{y_0}/K_v), \varphi_{y_0}, \mathrm{Fil}^\bullet_{y_0}) \\
\Longleftrightarrow\ &(H^*_{\mathrm{dR}}(X_{y_0}/K_v), \varphi_{y_0}, \nabla^*_{y,y_0}(\mathrm{Fil}^\bullet_y)) \cong (H^*_{\mathrm{dR}}(X_{y_0}/K_v), \varphi_{y_0}, \mathrm{Fil}^\bullet_{y_0}) \\
\Longleftrightarrow\ &\exists \sigma \in \mathrm{Aut}_{K_v}(H^*_{\mathrm{dR}}(X_{y_0}/K_v)) \text{ which commutes with } \varphi_{y_0} \text{ and sends } \nabla^*_{y,y_0}(\mathrm{Fil}^\bullet_y) \text{ to } \mathrm{Fil}^\bullet_{y_0} \\
\Longleftrightarrow\ &\nabla^*_{y,y_0}(\mathrm{Fil}^\bullet_y) \in Z(\varphi_{y_0}) \cdot \mathrm{Fil}^\bullet_{y_0}
\end{aligned}$$

Here $Z(\varphi_{y_0})$ denotes the set of all linear automorphisms of $H^*_{\mathrm{dR}}(X_{y_0}/K_v)$ which commutes with φ_{y_0}.

Note that the dimensions of $\mathrm{Fil}^\bullet_y$ do not rely on the choice of $y \in \Omega'_v$. Then we can classify different filtrations by constructing a period mapping

$$\Phi_v : \Omega'_v \to \mathcal{F}_v(K_v)$$

sending y to the filtration $\nabla_{y,y_0}(\mathrm{Fil}^\bullet_y)$, where $\mathcal{F}$ is a variety parameterized the flags of $H^*_{\mathrm{dR}}(X_{y_0}/K)$ and $\mathcal{F}_v = \mathcal{F} \otimes_K K_v$.

Consider the orbit $Z(\varphi) \cdot \mathrm{Fil}^\bullet_{y_0}$. If it has a smaller Zariski dimension than $\Phi_v(\Omega'_v)$, then we can deduce that the set of all y with an isomorphism

$$(H^*_{\mathrm{dR}}(X_y/K_v), \varphi_{y,v}, \mathrm{Fil}^\bullet_y) \cong (H^*_{\mathrm{dR}}(X_{y_0}/K_v), \varphi_{y_0,v}, \mathrm{Fil}^\bullet_{y_0})$$

has a smaller Zariski dimension than Ω'_v. Thus, we give a bound of the size of the fiber in 1.1.10. Now we consider the case Y is a curve. In this case, Ω_v is of dimension 1, then the fiber in 1.1.10 has dimension 0, which means that it is finite.

To be summarized, we have proved the following result:

Theorem 1.1.12. Let Y be a variety over a number field K. Assume that

(A) we are given a smooth proper K-morphism $f : X \to Y$, such that f extends to a smooth proper morphism $\mathcal{X} \to \mathcal{Y}$ between smooth models over $\mathcal{O} = \mathcal{O}_{K,S}$, where S is a finite set of places containing all Archimedean values of K and all ramified places of $K/\mathbb{Q}$;

(B) ρ_y is semisimple except for finitely many $y \in \mathcal{Y}(\mathcal{O})$;

(C) $\dim(Z(\varphi) \cdot \mathrm{Fil}^{\bullet}_{y_0}) < \dim(\overline{\Phi_v(\Omega'_v)})$.

where dim means the Zariski dimension. Then $\dim \Sigma_v < \dim \Omega'_v$.

In particular, if Y is a proper curve, then $\dim \Sigma_v = 0$, which means Σ_v is finite. Combining with 1.1.10 and 1.1.4 we then deduce that $Y(K)$ is finite.

1.1.2 The condition (C')

We get a beautiful theorem 1.1.12 in the last subsection, however, the condition (C) is a little complicated to understand. In this subsection, we will introduce a slightly stronger condition (C') but is easier to handle. We will prove the following statement.

Proposition 1.1.13. The condition (C) can be implied by the following condition (C').

$$\text{(C')} \ \left(\dim_{K_v} H^*_{\mathrm{dR}}(X_{y_0}/K_v)\right)^2 < \dim(\Gamma \cdot h_0^{\mathbb{C}}).$$

where Γ denotes the Zariski closure of the image of the monodromy representation

$$\overline{\mathrm{Im}(\pi_1(Y^{\mathrm{an}}_{\mathbb{C}}) \to \mathrm{GL}(H^*_{\mathrm{dR}}(X_{y_0,\mathbb{C}}/\mathbb{C})))}$$

and $h_0^{\mathbb{C}} = \Phi_{\mathbb{C}}(y_0)$. Indeed, $\dim_{K_v} H^*_{\mathrm{dR}}(X_{y_0}/K_v) \geq \dim(Z(\varphi) \cdot \mathrm{Fil}^{\bullet}_{y_0})$ and $\dim(\Gamma \cdot h_0^{\mathbb{C}}) \leq \dim(\overline{\Phi_v(\Omega'_v)})$.

First, one direction is easy:

Lemma 1.1.14. *Suppose that $\sigma : E \to E$ is a field automorphism of order e, with fixed field F. Let V be an E-vector space of dimension d, and $\varphi : V \to V$ a σ-semi-linear automorphism. Define the centralizer $Z(\varphi)$ of φ in $\mathrm{End}_E(V)$ via*

$$Z(\varphi) = \{f \in \mathrm{Aut}_E(V) : \ f \circ \varphi = \varphi \circ f\}$$

This is an F-vector space. Then

$$\dim_F (Z(\varphi)) = \dim_E (Z(\varphi^e))$$

where φ^e is now E-linear. In particular, $\dim_F (Z(\varphi)) \leq (\dim_E V)^2$.

Proof. Let $\bar{F}$ be an algebraic closure of F. Let Σ be the set of F-embedding $E \hookrightarrow \bar{F}$. Then $V \otimes_F \bar{F}$ can split as a decomposition

$$V \otimes_F \bar{F} = \bigoplus_{\tau \in \Sigma} V^{\tau}$$

where $V^{\tau} = V \otimes_{\tau} \bar{E}$ consists of elements $v \in V \otimes_F \bar{F}$ such that $ev = \tau(e)v$ for all $e \in E$. Indeed, every e can be identified as a matrix with $\tau(e)$ its eigenvectors. Moreover, φ extends to an $\bar{F}$-linear endomorphism $\bar{\varphi}$ of $V \otimes_F \bar{F}$, this endomorphism carries V^{τ} to $V^{\tau\sigma^{-1}}$.

Fix $\tau_0 \in \Sigma$; then

$$Z(\bar{\varphi}) \cong \text{ centralizer of } \bar{\varphi}^e \text{ on } V^{\tau_0}$$

By compare the dimensions we obtain the result. ■

Corollary 1.1.15. $(\dim_{K_v} H^*_{\mathrm{dR}}(X_{y_0}/K_v))^2 \geq \dim(Z(\varphi)\cdot \mathrm{Fil}^\bullet_{y_0})$.

Proof. Note that

$$\dim_{\mathrm{Zariski}}(Z(\varphi)\cdot \mathrm{Fil}^\bullet_{y_0}) \leq \dim_{\mathrm{Zariski}}(Z(\varphi^e)\cdot \mathrm{Fil}^\bullet_{y_0}) = \dim_{K_v}(Z(\varphi^e)) \leq (\dim_{K_v} H^*_{\mathrm{dR}}(X_{y_0}/K_v))^2$$

■

Let $\mathcal{F}$ be the flag K-variety associated to the Hodge filtration $F^\bullet H^*_{\mathrm{dR}}(X_{y_0}/K)$. Then we have period mappings

$$\Phi_{\mathbb{C}} : \Omega_{\mathbb{C}} \to \mathcal{F}_{\mathbb{C}}(\mathbb{C})$$

and

$$\Phi_v : \Omega'_v \to \mathcal{F}_v(K_v)$$

where $\Omega_{\mathbb{C}}$ is a contractible analytic disk containing y_0 in the complex analytic space $Y_{\mathbb{C}}(\mathbb{C})$. Recall that the mapping $\Phi_{\mathbb{C}}$ is $\mathbb{C}$-analytic.

The following lemma is the key result linking v-adic period and $\mathbb{C}$-period.

Lemma 1.1.16. *Suppose given power series $B_0, \cdots, B_N \in K[[z_1, \cdots, z_m]]$ such that all B_i are absolutely convergent, with no common zero, both in a v-adic and complex disks*

$$U_v = \{z : |z_i|_v < \epsilon\}, \quad U_{\mathbb{C}} = \{z : |z_i|_{\mathbb{C}} < \epsilon\}$$

Write

$$B_v : U_v \to \mathbb{P}^N_{K_v}$$
$$B_{\mathbb{C}} : U_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$$

for the corresponding maps sending z to $(B_0(z) : B_1(z) : \cdots : B_N(z))$.

Then there exists a K-subscheme $Z \subseteq \mathbb{P}^N$ whose base change to $\mathbb{C}$ (resp. K_v) gives the Zariski closure of $B_{\mathbb{C}}(U_{\mathbb{C}}) \subseteq \mathbb{P}^N_{\mathbb{C}}$ (resp. $B_v(U_v) \subseteq \mathbb{P}^N_{K_v}$). In particular, their Zariski closures have the same dimension.

Proof. We take the ideal I of Z to be the ideal generated by all homogeneous polynomials $Q \in K[x_0, \cdots, x_N]$ such that $Q(B_0, \cdots, B_N) \equiv 0$.

To verify the claim for K_v (the proof for $\mathbb{C}$ is identical) we just need to verify that if a homogeneous polynomial $Q_v \in K_v[x_0, \cdots, x_N]$ vanishes on $B_v(U_v)$ then Q_v lies in $I \otimes_K K_v$. Indeed, Q_v vanishes on $B_v(U_v)$ then $Q_v(B_0, \cdots, B_N) \equiv 0$. The coefficients of Q_v then form a solution of some linear equations with K-coefficients. Obviously the space of solutions on K_v is the base change of the space of solutions on K. ■

As a consequence of this lemma, we find that $\dim(\overline{\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})}) = \dim(\overline{\Phi_v(\Omega'_v)})$.

In the complex case, we can give a minimum scale of $\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})$ through the monodromy group.

Let $p : \tilde{Y}^{\mathrm{an}}_{\mathbb{C}} \to Y^{\mathrm{an}}_{\mathbb{C}}$ be the universal cover, i.e., it is a cover with $\tilde{Y}^{\mathrm{an}}_{\mathbb{C}}$ simply connected. Then by pulling back the local system $R^k\mathbb{C}$, we may similarly define a global period map

$$\tilde{\Phi}_{\mathbb{C}} : \tilde{Y}^{\mathrm{an}}_{\mathbb{C}} \to \mathcal{F}_{\mathbb{C}}(\mathbb{C})$$

This map is compatible with $\Phi_{\mathbb{C}}$ as the following commutative diagram:

$$
\begin{tikzcd}
p^{-1}(\Omega_{\mathbb{C}}) \arrow[r, hook] \arrow[d] & \tilde{Y}_{\mathbb{C}}^{\mathrm{an}} \arrow[d, "\tilde{\Phi}_{\mathbb{C}}"] \\
\Omega_{\mathbb{C}} \arrow[r, "\Phi_{\mathbb{C}}"'] & \mathcal{F}_{\mathbb{C}}(\mathbb{C})
\end{tikzcd}
$$

Note that the group $\pi_1(Y_{\mathbb{C}}(\mathbb{C}))$ acts on both sides of $\tilde{\Phi}_{\mathbb{C}}$, one is from the property of universal cover and the other is the monodromy action. Actually, it is proved by Griffiths that $\tilde{\Phi}_{\mathbb{C}}$ is compatible with these actions.

Then, by considering the Zariski closure, the inverse image of $\mathrm{Im}(\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}}))$ will be a subvariety of $\tilde{Y}$, and also, it contains many points in each "period". Then the set $\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})$ will has a very large Zariski closure. We now describe this idea explicitly.

Lemma 1.1.17. *Denote $h_0^{\mathbb{C}} = \Phi_{\mathbb{C}}(y_0)$. Choose $\tilde{y} \in p^{-1}(y_0)$, then $h_0^{\mathbb{C}} = \tilde{\Phi}_{\mathbb{C}}(\tilde{y})$. Let Γ denote the Zariski closure of the image of the monodromy representation $\overline{\mathrm{Im}(\pi_1(Y_{\mathbb{C}}(\mathbb{C})) \to \mathrm{GL}(H^*_{\mathrm{dR}}(X_{y_0,\mathbb{C}}/\mathbb{C})))}$. Then*

$$\Gamma \cdot h_0^{\mathbb{C}} \subseteq \textit{the Zriski closure of } \mathrm{Im}(\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}}))$$

Proof. For any subvariety $Z \subseteq \mathcal{F}_{\mathbb{C}}$, with $\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}}) \subseteq Z$, the preimage $\tilde{\Phi}_{\mathbb{C}}^{-1}(Z)$ is a complex-analytic subvariety containing $p^{-1}(\Omega_{\mathbb{C}})$, and thus all of $\tilde{Y}_{\mathbb{C}}^{\mathrm{an}}$ (see, for example, Theorem in [GR12] page 168). Therefore,

$$\pi_1(Y_{\mathbb{C}}(\mathbb{C})) \circlearrowleft h_0^{\mathbb{C}} = \tilde{\Phi}_{\mathbb{C}}(\pi_1(Y_{\mathbb{C}}(\mathbb{C})) \circlearrowleft \tilde{y}) \subseteq Z$$

Then the result follows immediately. ∎

The above two lemmas then deduce that

Lemma 1.1.18. *The dimension of the Zariski closure of $\Phi_v(\Omega'_v)$ is at least the complex dimension of $\Gamma \cdot h_0^{\mathbb{C}}$.*

In particular, if $\mathcal{F}_v^{\mathrm{bad}} \subseteq \mathcal{F}_v$ is a Zariski-closed subset of dimension less than the complex dimension of $\Gamma \cdot h_0^{\mathbb{C}}$, then $\Phi_v^{-1}(\mathcal{F}_v^{\mathrm{bad}})$ is contained in a proper K_v-analytic subset of Ω'_v, by which we mean a subset killed by a v-adic series converging absolutely on Ω'_v.

Then we can re-write 1.1.12 as following:

Proposition 1.1.19. Let Y be a variety over a number field K. Assume that

(A) we are given a smooth proper K-morphism $f : X \to Y$, such that f extends to a smooth proper morphism $\mathcal{X} \to \mathcal{Y}$ between smooth models over $\mathcal{O} = \mathcal{O}_{K,S}$, where S is a finite set of places containing all Archimedean values of K and all ramified places of $K/\mathbb{Q}$;

(B) ρ_y is semisimple except for finitely many $y \in \mathcal{Y}(\mathcal{O})$;

(C') $\left(\dim_{K_v} H^*_{\mathrm{dR}}(X_{y_0}/K_v)\right)^2 < \dim(\Gamma \cdot h_0^{\mathbb{C}})$.

Then $\dim \Sigma_v < \dim \Omega'_v$. In particular, if Y is a proper curve, then $\dim \Sigma_v = 0$, which means Σ_v is finite. Combining with 1.1.10 and 1.1.4 we then deduce that $Y(K)$ is finite.

1.1.3 The modified version

1. Change the family. Unfortunately, sometimes $\dim_{K_v} H^*_{\mathrm{dR}}(X_{y_0}/K_v)$ is too big that (****) may not hold true. We want to slightly change the method to lessen the dimension. Indeed, we can control the dimension of $Z(\varphi)$ by considering a large extension $L_w \hookleftarrow K_v \hookleftarrow \mathbb{Q}_p$. We will consider a new smooth proper K-family $X \to Y' \xrightarrow{\pi} Y$, where π is finite and étale.

To continue illustrating how the modification works, we first need to change the condition (A). We replace it with the following condition:

> *(A') there is a family of smooth proper K-morphisms $X \to Y' \xrightarrow{\pi} Y$ with π a finite étale cover, such that they can extend to smooth proper $\mathcal{O}$-morphisms $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$ between $\mathcal{O}$-models with $\mathcal{Y}' \to \mathcal{Y}$ finite and étale. Further, we assume that fix a number $* \geq 0$, the function $z \mapsto \dim_{\kappa(z)} \left(H^*_{\mathrm{dR}}(X_z/\kappa(z))\right)$ is constant on $z \in \pi^{-1}(Y)$.*

From the morphism

$$X_y \to (Y')_y = \mathrm{Spec}\left(\prod_{z\in\pi^{-1}(y)} \kappa(z)\right)$$

$H^*_{\mathrm{dR}}(X_y/K) = \bigoplus_{z\in\pi^{-1}(y)} H^*_{\mathrm{dR}}(X_z/\kappa(z))$ equips with a $\prod_{z\in\pi^{-1}(y)}\kappa(z)$-module structure, which respects the split. Thus, as a $E(y) \triangleq \prod_{z\in\pi^{-1}(y)}\kappa(z)$-module, $H^*_{\mathrm{dR}}(X_y/K)$ is free, since every $H^*(X_z/\kappa(z))$ is a free $\kappa(z)$-vector space with the same dimension.

In particular, for a place $v|p$, $H^*_{\mathrm{dR}}(X_y/K_v) = \bigoplus_{z\in\pi^{-1}(y),w|v} H^*_{\mathrm{dR}}(X_z/\kappa(z)_w)$ equips with a free $E(y)\otimes_K K_v = \prod_{z\in\pi^{-1}(y),w|v}\kappa(z_0)_w$-module structure.

The p-adic parallel transportation $\nabla^0_{y,y_0,Y'\to Y}$ for $Y' \to Y$ implies that there is a morphism between K_v-algebras

$$\prod_{z\in\pi^{-1}(y),w|v} \kappa(z)_w \cong \prod_{z_0\in\pi^{-1}(y_0),w_0|v} \kappa(z_0)_{w_0}$$

since ∇ respects the cup product, which gives H^0_{dR} a K_v-algebra structure. In particular, this is a ring-isomorphism. Then there is a bijection between multisets $\{\kappa(z)_w : z \in \pi^{-1}(z), w|v\}$ and $\{\kappa(z_0)_{w_0} : z_0 \in \pi^{-1}(z_0), w_0|v\}$ (by considering the quotients through maximal ideals) such that ∇^0_{y,y_0} sends $\kappa(z)_w$ to $\kappa(z_0)_{w_0}$. Then there is an isomorphism of fields $\kappa(z)_w \cong \kappa(z_0)_{w_0}$ which preserves K_v.

Remark 1.1.20. *We could not imply the isomorphism $\kappa(z) \cong \kappa(z_0)$ from the correspondence*

$$\{\kappa(z)_w\} \leftrightarrow \{\kappa(z_0)_{w_0}\}$$

(At least I do not know how to deduce this.) However, note that $\pi^{-1}(y)(\bar{K})$ admits a S-model, then $\kappa(z)$ is unramified outside S. In particular, the set of $\kappa(z)$ is finite under isomorphism.

The p-adic parallel transportation for $X \to Y$ provides a K_v-linear isomorphism

$$H^*_{\mathrm{dR}}(X_y/K_v) \to H^*_{\mathrm{dR}}(X_{y_0}/K_v)$$

This is compatible with the isomorphism between K_v-algebras

$$\begin{array}{lccc}\nabla_{y,y_0,X\to Y}: & \bigoplus_{z\in\pi^{-1}(y),w|v} H^*_{\mathrm{dR}}(X_z/\kappa(z)_w) & \longrightarrow & \bigoplus_{z_0\in\pi^{-1}(y_0),w_0|v} H^*_{\mathrm{dR}}(X_{z_0}/\kappa(z_0)_{w_0}) \\ & \uparrow & & \uparrow \\ \nabla_{y,y_0,Y\to Y'}: & H^0_{\mathrm{dR}}((Y')_y/K_v) & \longrightarrow & H^0_{\mathrm{dR}}((Y')_{y_0}/K_v) \\ & \uparrow\sim & & \uparrow\sim \\ & \bigoplus_{z\in\pi^{-1}(y),w|v} \kappa(z)_w & \longrightarrow & \bigoplus_{z_0\in\pi^{-1}(y_0),w_0|v} \kappa(z_0)_w\end{array}$$

which also respects the split (because ∇ is functorial). In particular, there is a parallel transportation

$$\begin{array}{ccc} H^*_{\mathrm{dR}}(X_z/\kappa(z)_w) & \xrightarrow{\nabla_{(z,w),(z_0,w)}} & H^*_{\mathrm{dR}}(X_{z_0}/\kappa(z_0)_{w_0}) \\ \uparrow & & \uparrow \\ \kappa(z)_w & \xrightarrow{\nabla_{y,y_0,Y\to Y'}} & \kappa(z_0)_{w_0}\end{array}$$

if we identify $\kappa(z)_w$ with $\kappa(z_0)_{w_0}$ through ∇, the above line is an isomorphism as vector spaces.

Similarly, we can construct the period mappings

$$\Phi_{\mathbb{C}}:\Omega_{\mathbb{C}}\to\mathcal{F}_{\mathbb{C}}(\mathbb{C})$$

$$\Phi_v:\Omega'_v\to\mathcal{F}_v(K_v)$$

where $\mathcal{F}$ parameterizes the filtrations with the same dimensions with $\bigoplus_{z_0\in\pi^{-1}(y_0),w_0|v}\mathrm{Fil}^{\bullet}_{z_0,w_0}$. Here $\mathrm{Fil}^{\bullet}_{z_0,w_0}$ denotes the Hodge filtration of $H^*_{\mathrm{dR}}(X_{z_0}/\kappa(z_0)_{w_0})$. Since $\nabla_{y,y_0,X\to Y}$ and $\nabla_{\mathbb{C}}$ both respect the decomposition, we can find that $\Phi_{\mathbb{C}}$ (resp. Φ_v) actually has image in $\bigoplus_{z_0\in\pi^{-1}(y_0)}\mathcal{F}_{\mathbb{C},z_0}(\mathbb{C})$ (resp. $\bigoplus_{z_0\in\pi^{-1}(y_0),w_0|v}\left(\mathrm{Res}^{\kappa(z_0)_{w_0}}_{K_v}\mathcal{F}_{z_0,w_0}\right)(K_v)$). Here $\mathcal{F}_{\mathbb{C},z_0}$ (resp. $\mathcal{F}_{z_0,w_0}$) denotes the flag variety corresponding to the Hodge filtration of $H^*_{\mathrm{dR}}((X_z)_{\mathbb{C}}/\mathbb{C})$ (resp. $H^*_{\mathrm{dR}}(X_z/\kappa(z_0)_{w_0})$), Res means Weil restriction.

2. The condition (B'). For the finiteness of $\mathcal{Y}(\mathcal{O})$, or $\Omega_v=\{y\in\mathcal{Y}(\mathcal{O}):y\equiv y_0(\mathrm{mod}\ v)\}$, we will use representations $\rho_z=H^*_{\text{ét}}\left((X_z)_{\bar{\kappa(z)}},\mathbb{Q}_p\right)$ for some $z\in\pi^{-1}(y)$ rather than ρ_y. We also use a new version of the second assumption here.

(B') for almost all $y\in\Omega_v$, ρ_z is semi-simple for ***at least one*** *$z\in\pi^{-1}(y)$.*

Remark 1.1.21. *In some cases, like the S-unit equation we will discuss in chapter 4, we can use a stronger condition to significantly reduce some technical steps.*

(B") for almost all $z\in\pi^{-1}(\mathcal{Y}(\mathcal{O}))$, ρ_z is semi-simple.

However, in the original paper by Lawrence and Venkatesh, their method on Mordell conjecture could not imply such a strong result (if we avoid the using of Tate conjecture), so we only use condition (B') here.

We need some reductions here! Comparing to section 1.1.1, we face some problems to continue the discussion.

- Note that ρ_z is a representation of $G_{\kappa(z)}$. To conclude the finiteness of ρ_z (using Faltings theorem 1.1.5), we should fix one number field.
- Although we can require that $\kappa(z) \cong \kappa(z_0)$, we can not guarantee that $\nabla_{y,y_0,Y\to Y'}$ sends $\kappa(z)_w$ to $\kappa(z_0)_{w_0}$, since there are examples that non-isomorphic global fields induce isomorphic local fields.

To fix the above problems, recall 1.1.20 the set of $\kappa(z)$ is finite under isomorphism.

For almost all $y \in \Omega_v$, we fix one $z \in \pi^{-1}(y)$ satisfying condition (B'). We attach y the tuple $(\kappa(z), \cdots, \kappa(z)_w, \cdots)$ which consists of the global field $\kappa(z)$ and all its induced local fields $\kappa(z)_w$ for $w|v$. This tuple has finite coordinates. We say two tuples $(\kappa(z), \cdots, \kappa(z)_w, \cdots)$ and $(\kappa(z_0), \cdots, \kappa(z_0)_{w_0}, \cdots)$ are isomorphic if $\kappa(z) \cong \kappa(z_0)$ and $\kappa(z)_w$ is sent to $\kappa(z_0)_{w_0}$ by $\nabla_{y,y_0,Y'\to Y}$. Since $(\kappa(z), \cdots, \kappa(z)_w, \cdots)$ has finite choices, we can split Ω_v into a finite union $\Omega_v = \cup_{\text{finite}}\Xi$.

3. Conclude the Main Theorem. By Faltings theorem 1.1.5 and (B'), under isomorphism, for $y \in \Xi$, ρ_z takes values in a finite set under isomorphism. Then we hope to show that $y \mapsto \rho_z$ has finite fibers.

Now we consider the functor

$$y \mapsto \rho_z \mapsto \rho_{z,w}$$

The corresponding filtered φ-module can be described as

$$(H^*_{\mathrm{dR}}(X_z/\kappa(z)_w), \varphi, \mathrm{Fil}^\bullet_{z,w})$$

Then we consider the projection of the period mapping

$$\Phi_{z,w} : \Omega'_v \to \left(\mathrm{Res}^{\kappa(z)_w}_{K_v} \mathcal{F}_{z,w}\right)(K_v)$$

where $\mathcal{F}_{z,w}$ parameters the subspaces of $H^*_{\mathrm{dR}}(X_z/\kappa(z)_w)$. The remaining arguments are the same, following the steps in section 1.1.2, we can also claim the following condition.

$$\text{(C”)}\ \dim(\Gamma \cdot h_0^{\mathbb{C}}) > \left(\mathrm{rank}_{E(y)}(H^*_{\mathrm{dR}}(X_y/K_v))\right)^2 = \left(\dim_{\kappa(z)_w}(H^*_{\mathrm{dR}}(X_z/\kappa(z)_w))\right)^2$$

Now 1.1.12 turns to be the following form:

Theorem 1.1.22. Let Y be a variety over a number field K. Assume that

(A') there is a family of smooth proper K-morphisms $X \to Y' \xrightarrow{\pi} Y$ with π a finite étale cover, such that they can extend to smooth proper $\mathcal{O}$-morphisms $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$ between $\mathcal{O}$-models with $\mathcal{Y}' \to \mathcal{Y}$ smooth and étale. Further, we assume that fix a number $* \geq 0$, the function $z \mapsto \dim_{\kappa(z)}\left(H^*_{\mathrm{dR}}(X_z/\kappa(z))\right)$ is constant on $z \in \pi^{-1}(Y)$.;

(B') for almost all $y \in Y(K)$, ρ_z is semi-simple at least for one $z \in \pi^{-1}(y)$.

(C”) $\dim(\Gamma \cdot h_0^{\mathbb{C}}) > \left(\dim_{\kappa(z)_w}\left(H^*_{\mathrm{dR}}(X_z/\kappa(z)_w)\right)\right)^2$.

Then $\dim \Xi < \dim \Omega'_v$. In particular, if Y is a proper curve, then $\dim \Xi = 0$, which means Σ_v is finite, or equivalently, $Y(K)$ is finite.

It is easy to see that the advantage of 1.1.22 comparing with 1.1.19 is that we decrease the rank of $H^*_{\mathrm{dR}}(X_{y_0}/K_v)$ by enlarge the base ring. However, where there is gain, there is loss, this result requires a more complicated construction of families.

Also, what we should highlight here is that the degree $[\kappa(z) : K]$ is an important variation in our future analyze. One obvious reason is that this number directly relates to the dimension of $\mathrm{rank}_{E(y)}(H^*_{\mathrm{dR}}(X_{y_0}/K_v))$ in some cases, see chapter 4 for why we require $[\kappa(z) : K] > 4$. The other reason is that we can use this to cope with the non-semi-simple points.

For the sake of simplifying our main theorem, we slightly introduce the special family we will use in the proof of the Mordell conjecture.

Example 1. If we assume that $X \to Y'$ is an Abelian scheme with polarization and relative dimension d, then by Tate conjecture ρ_z is semi-simple. (We should highlight here again that we will not use the Tate conjecture to prove the Mordell conjecture, but currently we can convince ourselves that the semi-simplicity condition is true.)

Also, to simplify the conditions further, we assume that $X \to Y$ has full monodromy, which means $\Gamma \supseteq \mathrm{Sp}(H^*_{\mathrm{dR}}(X_y/\mathbb{C}), w)$. Here w means the symplectic pairing defined by the polarization. Then $\Gamma \cdot h_0^{\mathbb{C}}$ take values in all $\left(\mathrm{Res}_{K_v}^{\kappa(z_0)_{w_0}} \mathrm{LGr}_{z_0,w_0}\right)(K_v) \subseteq \left(\mathrm{Res}_{K_v}^{\kappa(z_0)_{w_0}} \mathbf{Gr}_{z_0,w_0}\right)(K_v)$. Here LGr is the Lagrangian Grassmannian and Gr is the usual Grassmannian. Hence $\dim(\Gamma \cdot h_0^{\mathbb{C}}) \geq [\kappa(z_0)_{w_0} : K_v] \cdot \dfrac{d(d+1)}{2}$.

Note that $\dim_{\kappa(z)_w}\left(H^*_{\mathrm{dR}}(X_z/\kappa(z)_w)\right) = d^2$, then condition (C”) holds if and only if $[\kappa(z)_w : K_v] \geq 8$.

Chapter 2

p-adic parallel transportation

In this chapter, we will illustrate the construction of the parallel transportation

$$\nabla : H^*_{\mathrm{dR}}(X_y/K_v) \to H^*_{\mathrm{dR}}(X_{y_0}/K_v)$$

omitted in the last chapter. One can skip this chapter by admitting its existence.

2.1 Some results in complex analytic case

We first introduce some famous results on complex analytic spaces.

2.1.1 The relative de Rham cohomology

Definition 2.1.1. Consider a smooth projective morphism $\pi : X \to Y$ of smooth projective complex analytic spaces. The relative de Rham cohomology $H^q_{\mathrm{dR}}(X/Y)$ is defined by the hypercohomology

$$H^q_{\mathrm{dR}}(X/Y) = \mathbb{R}^q\pi_*\Omega^\bullet_{X/Y}$$

where $\Omega^\bullet_{X/Y}$ is the relative holomorphic de Rham complex

$$\Omega^\bullet_{X/Y} : \mathcal{O}_X \to \Omega^1_{X/Y} \to \Omega^2_{X/Y} \to \cdots$$

Definition 2.1.2. Let $\Omega^{\bullet\geq i}_{X/Y}$ be the complex $\Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y} \to \cdots$. The Hodge filtration $F^\bullet H^q_{\mathrm{dR}}(X/Y)$ is defined as

$$F^i H^q_{\mathrm{dR}}(X/Y) = \mathrm{Im}(\mathbb{R}^q\pi_*\Omega^{\bullet\geq i}_{X/Y} \to \mathbb{R}^q\pi_*\Omega^\bullet_{X/Y})$$

The associated spectral sequence

$$E_1^{p,q} = R^q\pi_*\Omega^p_{X/Y} \Rightarrow \mathbb{R}^{p+q}\pi_*\Omega^\bullet_{X/Y} = H^{p+q}_{\mathrm{dR}}(X/Y)$$

is called the relative Fröhlicher spectral sequence, this spectral sequence degenerates at E_1-page. Here, the left hand means the cohomology of an injective resolution of $\Omega^q_{X/Y}$ with respect to π_*, or equivalently the hypercohomology of $\cdots \to 0 \to \Omega^q_{X/Y} \to 0 \to \cdots$ with respect to π_*. Deligne ([Del68]) has proved that this spectral sequence degenerates at E_1-page and $\pi_*\Omega^p_{X/Y}$ are vector bundles over Y, in particular, $H^\bullet_{\mathrm{dR}}(X/Y)$ are vector bundles over Y.

Corollary 2.1.3 (Comparison theorem)**.** The relative holomorphic Poincaré lemma tells us that there is a quasi-isomorphism $\pi^{-1}\mathcal{O}_Y \to \Omega^\bullet_{X/Y}$. Then there is an isomorphism

$$R^q\pi_*\left(\pi^{-1}\mathcal{O}_Y\right) \cong H^q_{\mathrm{dR}}(X/Y)$$

In particular, if $Y = \mathrm{Spec}(\mathbb{C})$, it turns to be the following isomorphism

$$H^q(X, \mathbb{C}) \cong H^q_{\mathrm{dR}}(X/\mathbb{C})$$

2.1.2 The analytic Riemann-Hilbert correspondence.

The main reference of this section is [Sza09] chapter 2.

Let Y be a topological space. We assume it is connected, locally path-connected and locally simply connected, so it has a nice universal cover.

Definition 2.1.4. Let S be a topological space with the discrete topology. Define a sheaf $\mathcal{F}_S$ on Y by sending an open subset $U \subseteq Y$ to all continuous maps $U \to S$. This sheaf is called the constant sheaf on Y.

Theorem 2.1.5 ([Sza09] Theorem 2.5.9, the first equivalence)**.** Let $p : X \to Y$ be a cover of topological spaces. Define a presheaf $\mathcal{F}_X$ by sending $U \subseteq Y$ to all sections $U \to X$ of $p : p^{-1}(U) \to U$. Then this sheaf is a locally constant sheaf.

Further, this functor induces an equivalence between the category of covers of Y and locally constant sheaves on Y.

Definition 2.1.6. Given a cover $p : X \to Y$. Fix a point $y \in Y$ and let $\gamma \in \pi_1(Y, y)$ be a loop of y. By the lifting lemma ([Sza09] Lemma 2.3.2), for any point $x = p^{-1}(y)$, there is a unique path $\gamma_x \in X$ lifting γ with $\gamma_x(0) = x$. Define the monodromy action of $\pi_1(Y, y)$ on $p^{-1}(y)$ as $\gamma(x) = \gamma_x(1)$.

Proposition 2.1.7 ([Sza09] Theorem 2.3.4, the second equivalence)**.** There is an equivalence of the category of covers of Y and the category of left $\pi_1(Y, y)$-sets by sending $p : X \to Y$ to $p^{-1}(y)$.

Definition 2.1.8. A complex local system $\mathcal{F}$ on Y is a locally constant sheaf on y whose stalks are finitely-dimensional complex vector spaces.

Corollary 2.1.9 ([Sza09] Corollary 2.6.2, the third equivalence)**.** Fix a base point $y \in Y$. If we have a local system $\mathcal{F}$, there is a representation

$$\pi_1(Y, y) \to \mathrm{GL}(\mathcal{F}_y)$$

The functor defines an equivalence between isomorphism classes of local systems and isomorphism classes of representations of the fundamental group of $\pi_1(Y, y)$.

Corollary 2.1.10. If $\mathcal{F}$ is a locally constant sheaf on a simply-connected space, then $\mathcal{F}$ is constant.

We will use the following result in chapter 4.

Corollary 2.1.11. Let (V, ρ) be the representation of $\pi_1(X, x)$ that corresponds to a local system $\mathcal{F}$. Then $h_*\mathcal{F}$ is a local system over Y and it corresponds to the representation $\mathrm{Ind}_{\pi_1(X,x)}^{\pi_1(Y,y=p(x))}(V, \rho)$.

Proof. Since h^* and h_* are adjoint functors, the result follows from that Ind and Res are adjoint functors. ■

Definition 2.1.12. Given a morphism $\pi : X \to Y$ of complex analytic spaces. Let E be a holomorphic vector bundle on X. A relative connection ∇ on E is a $\pi^{-1}\mathcal{O}_Y$-linear map

$$\nabla : E \to E \otimes_{\mathcal{O}_X} \Omega^1_{X/Y}$$

that satisfies the Leibniz rule:

$$\nabla(f\sigma) = f\nabla(\sigma) + \sigma \otimes df$$

We can further differentiate it $\nabla : E \otimes \Omega_{X/Y} \to E \otimes \bigwedge^2 \Omega_{X/Y}$ by the rule

$$\nabla(\sigma \otimes \alpha) = \nabla\sigma \wedge \alpha + \sigma \otimes d\alpha$$

We also define the curvature of a connection to be the composite $\theta = \nabla \circ \nabla$. We say that a connection is flat or integrable if it has curvature 0.

Example 2. For any complex local system H on $Y/\mathbb{C}$, we can associate a holomorphic vector bundle $\mathcal{H} = H \otimes_{\mathbb{C}} \mathcal{O}_Y$ over Y. Define a connection

$$\nabla : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_Y} \Omega_Y$$

in the following way. For any $\sigma = \sum a_i\sigma_i$, set

$$\nabla\sigma = \sum \sigma_i \otimes da_i$$

Note that H is equal to $\mathrm{Ker}(\nabla)$.

Theorem 2.1.13 ([Sza09] Proposition 2.7.5, Riemann-Hilbert correspondence)**.** The above construction produces a bijection between isomorphism classes of complex local systems and isomorphism classes of holomorphic vector bundles equipped with a flat connection.

Remark 2.1.14. *By the projection lemma for ringed spaces ([Sta23, Tag 01E8])* $(X, \mathbb{C})$ *and* $(Y, \mathbb{C})$*, we find that*

$$\mathcal{O}_Y \otimes_{\mathbb{C}} R^q\pi_*\mathbb{C} \cong R^q\pi_*\left(\pi^*\mathcal{O}_Y \otimes_{\mathbb{C}} \mathbb{C}\right) = R^q\pi_*\left(\pi^{-1}\mathcal{O}_Y \otimes_{\pi^{-1}\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}\right) = R^q\pi_*\left(\pi^{-1}\mathcal{O}_Y\right)$$

Thus $R^q\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_Y \cong H^q_{\mathrm{dR}}(X/Y)$*.*

We will see later 2.1.17, the sheaf $R^q\pi_*\mathbb{C}$ *is a complex local system.*

2.1.3 Analytic Gauss-Manin connection

We first verify that $R^q\pi_*\mathbb{C}$ is a complex local system.

Definition 2.1.15. A differentiable map between differentiable manifolds is called proper if the inverse image of compact set is compact.

A differentiable map between differentiable manifolds is called submersion if the differential is everywhere surjective.

Theorem 2.1.16 ([Voi02]Theorem 9.3, Ehresmann theorem)**.** If $\varphi : X \to Y$ is a proper surjective submersion between two differentiable manifolds, where Y is contractible with base point 0. Then there exists a diffeomorphism

$$X \cong X_0 \times Y$$

over Y.

Corollary 2.1.17. Let $\pi : X \to Y$ be a proper submersion between differentiable manifolds. Consider the sheaf $R^q\pi_*\mathbb{C}$ on Y. This sheaf $R^q\pi_*\mathbb{C}$ is a local system.

Proof. By Ehresmann theorem, X is isomorphic to $X_0 \times U$ for a contractible subset $U \subseteq Y$ containing 0, where $X_0 = \pi^{-1}(0)$.

The Proposition III. 8.1 of [Har13] tells us that $R^q\pi_*\mathbb{C}$ is the sheaf associated to the presheaf

$$V \mapsto H^i(\pi^{-1}(V), \mathbb{C}|_{f^{-1}(V)})$$

on Y.

Note that for any open subset $V \subseteq U$ we have $\pi^{-1}(V) \cong V \times X_0$, as well as V is contractible,

$$H^i(\pi^{-1}(V), \mathbb{C}|_{f^{-1}(V)}) \cong H^i(X_0, \mathbb{C})$$

Therefore, $R^q\pi_*\mathbb{C}$ is a local system, isomorphic to the constant sheaf with stalk $H^k(X_0, \mathbb{C})$ around 0. ■

Definition 2.1.18. By 2.1.14 the holomorphic vector bundle defined by $R^q\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_Y$ is exactly $H^q_{\mathrm{dR}}(X/Y)$. The associated flat connection

$$\nabla : H^q_{\mathrm{dR}}(X/Y) \to H^q_{\mathrm{dR}}(X/Y) \otimes_{\mathcal{O}_Y} \Omega_Y$$

is called the Gauss-Manin connection.

Fix $y_0 \in Y$ and a simply-connected open neighborhood $\Omega \subseteq Y$ of y_0. Consider the fiber at $y \in \Omega$. From the proof of 2.1.17, we have an isomorphism

$$H^q(X_y, \mathbb{C}) \cong H^q(X_{y_0}, \mathbb{C})$$

Through the comparison theorem 2.1.3, this isomorphism becomes

$$\nabla_{y_0,y} : H^q_{\mathrm{dR}}(X_{y_0}/\mathbb{C}) \to H^q_{\mathrm{dR}}(X_y/\mathbb{C})$$

which called the analytic parallel transportation.

We can describe this isomorphism more explicitly.

Choose a local basis $e_1, \cdots, e_r$ (resp. $dz^1, dz^2, \cdots, dz^n$) of the vector bundle $\left(H^q_{\mathrm{dR}}(X/Y)\right)|_\Omega$ (resp. $\Omega^1_{X/Y}$). Assume that ∇ sends e_i to $\sum_{j,k} \Gamma^j_{i,k} e_j \otimes dz^k$. Then for any $f = \sum f_i e_i$, $f_i \in \mathcal{O}_\Omega$,

$$\nabla f = 0 \iff \frac{\partial f_i}{\partial z_k} + \sum_j \Gamma^j_{i,k} f_j = 0 \qquad (*)$$

for all i, k.

For any $\alpha \in (\mathrm{Ker}(\nabla))_{y_0}$, there exists a unique formal solution f of the equations above such that $\alpha \otimes 1 = f$. Recall that $\mathrm{Ker}(\nabla)$ is exactly the constant sheaf $R^q\pi_*\mathbb{C}$, then $\nabla_{y_0,y}$ is defined by $\alpha \mapsto f|_y$.

2.2 Algebraic case

2.2.1 Algebraic relative de Rham cohomology

We can similar define the algebraic relative de Rham cohomology for any smooth proper morphism $\pi : X \to Y$ of S-schemes. The only difference is that we take the relative Kähler differential sheaves $\Omega^\bullet_{X/Y}$ here. Similarly, we have the Hodge filtrations and the relative Fröhlicher spectral sequence

$$F^i H^q_{\mathrm{dR}}(X/Y) = \mathrm{Im}(\mathbb{R}^q\pi_*\Omega^{\bullet\geq i}_{X/Y} \to \mathbb{R}^q\pi_*\Omega^\bullet_{X/Y})$$

$$E_1^{p,q} = R^q\pi_*\Omega^p_{X/Y} \Rightarrow \mathbb{R}^{p+q}\pi_*\Omega^\bullet_{X/Y} = H^{p+q}_{\mathrm{dR}}(X/Y)$$

Theorem 2.2.1 ([Del68] Theorem 5.5)**.** If π is a smooth proper morphism with char $(Y) = 0$. Then the relative Fröhlicher spectral sequence for X over Y degenerates at E_1-page, and the sheaves $R^q\pi_*\Omega^p_{X/Y}$ are vector bundles over Y. In particular, every $H^n_{\mathrm{dR}}(X/Y)$ is a vector bundle.

2.2.2 The algebraic Gauss-Manin connection

Similarly we have a definition of relative connections in the algebraic case. The only difference is that we take E to be the vector bundles.

In the algebraic case, we have a topology with bad connectivity (the Zariski topology). As a result, we could not construct the Gauss-Manin connection with good properties through local systems.

We will follow the approach in [KO68], which uses double complex to manifest this information.

Let R be any commutative ring. Let $\pi : X \to Y$ be a smooth, proper morphism between smooth R-schemes.

Recall that there is an exact sequence

$$0 \to \pi^*(\Omega^1_{Y/R}) \to \Omega^1_{X/R} \to \Omega^1_{X/Y} \to 0 \quad (**)$$

Let

$$F^i\Omega^\bullet_{X/R} = \mathrm{Im}(\Omega^{\bullet-i}_{X/R} \otimes_{\mathcal{O}_X} \pi^*(\Omega^i_{Y/R}) \to \Omega^\bullet_{X/R})$$

Now we compute the graded piece $G_i = F^i/F^{i+1}$. Since π is smooth, the exact sequence (**) locally split (since they are all locally free) and then locally we have

$$\Omega^n_{X/R} = \bigwedge^n \Omega^1_{X/R} \cong \bigwedge^p(\pi^*(\Omega^1_{Y/R}) \oplus \Omega^1_{X/Y}) = \bigoplus_{0\leq i\leq p} (\pi^*(\Omega^i_{Y/R}) \otimes \Omega^{n-i}_{X/Y})$$

Thus

$$\begin{aligned} F^i\Omega^n_{X/R} = &\text{ the image of } \Omega^{n-i}_{X/R} \otimes_{\mathcal{O}_X} \pi^*(\Omega^i_{Y/R}) \text{ under the wedge product} \\ = &\bigoplus_{j\geq i}(\pi^*(\Omega^j_{Y/R}) \otimes \Omega^{n-j}_{X/Y}) \end{aligned}$$

Hence $G_i = F^{i+1}/F^i = \pi^*(\Omega^i_{Y/R}) \otimes_{\mathcal{O}_X} \Omega^{n-i}_{X/Y}$.

Let $E_i^{p,q}$ be the spectral sequence associated to this filtration. The first page is

$$E_1^{p,q} = R^{p+q}\pi_*(G_i\Omega^\bullet_{X/R}) = R^{p+q}\pi_*(\pi^*(\Omega^p_{Y/R}) \otimes_{\mathcal{O}_X} \Omega^{\bullet-q}_{X/Y}) = \Omega^p_{Y/R} \otimes_{\mathcal{O}_Y} R^q\pi_*(\Omega^\bullet_{X/Y})$$

The last equality follows from the projection formula. Then the first page gives us a sequence

$$0 \to E_1^{0,n} = H^n_{\mathrm{dR}}(X/Y) \xrightarrow{d} E_1^{1,n} = \Omega^1_{Y/R} \otimes_{\mathcal{O}_Y} H^n_{\mathrm{dR}}(X/Y) \xrightarrow{d} \cdots$$

This d is called the algebraic Gauss-Manin connection.

2.3 Formal parallel transportation

For now, let K be a number field. Let Y be a smooth K-scheme of dimension n and $\pi : X \to Y$ a smooth proper morphism over K, which extends to $\tilde{\pi} : \mathcal{X} \to \mathcal{Y}$ over $\mathcal{O} = \mathcal{O}_{K,S}$ with $\mathcal{Y}$ smooth over $\mathcal{O}$. We have the Gauss-Manin connection on $\mathcal{Y}$

$$\nabla : H^q_{\mathrm{dR}}(\mathcal{X}/\mathcal{Y}) \to \Omega^1_{\mathcal{Y}/\mathcal{O}} \otimes_{\mathcal{O}_{\mathcal{Y}}} H^q_{\mathrm{dR}}(\mathcal{X}/\mathcal{Y})$$

We want to define an analogue parallel transportation through the local coordinates.

By enlarging S, we may assume that all sheaves $H^q_{\mathrm{dR}}(\mathcal{X}/\mathcal{Y})$ are locally free of finite rank over $\mathcal{Y}$.

For some $y_0 \in \mathcal{Y}(\mathcal{O})$, fix a local basis $\{v_1, \cdots, v_r\}$ of $H^q_{\mathrm{dR}}(\mathcal{X}/\mathcal{Y})$. We may write $\nabla v_j = \sum_i A^i_j v_i$, where A^i_j are local sections of $\Omega^1_{\mathcal{Y}/\mathcal{O}}$ near y_0. Then for any $f = \sum_i f_i v_i$ for $f_i \in \mathcal{O}$,

$$\nabla f = 0 \iff d(f_i) = -\sum_j A^i_j f_j, \ \forall i$$

Now we want to locally write A^i_j into the form $\sum \Gamma^{jk}_i dz_k$. Note that the formal solution of $(*)$ need to be convergent, we may need to discuss in a complete field K_v. Thus, let us first take a look at the ring containing these coefficients Γ^{jk}_i, which firstly should be in $\mathcal{O}_{\mathcal{Y},y_0}$.

From now, we use p to denote the rational prime number below v. Let v be a place of K such that

- $p > 2$.
- v is unramified in $K/\mathbb{Q}$.
- Any place v' above p is not in S.

$$\begin{tikzcd}
\bar{\mathcal{X}}_v \arrow[r] & \bar{\mathcal{Y}}_v & \mathrm{Spec}(\mathbb{F}_v) \arrow[d] \arrow[ddl, "\overline{y_0}"'] \\
\mathcal{X}_v \arrow[r] & \mathcal{Y}_v & \mathrm{Spec}(\mathcal{O}_v) \arrow[dl, "y_0"] \\
\mathcal{X} \arrow[r] & \mathcal{Y} & \mathrm{Spec}(\mathcal{O}) \\
X \arrow[r] & Y \arrow[u] & \mathrm{Spec}(K) \arrow[u]
\end{tikzcd}$$

Let $\overline{y_0} \in \mathcal{Y}(\mathbb{F}_v)$ be the reduction of y_0, which corresponds to a point $\bar{y}_0 \in \mathcal{Y}$ with an inclusion $k(\bar{y}_0) \hookrightarrow \mathbb{F}_v$. Note that the morphism $y_0 : \mathrm{Spec}(\mathcal{O}) \to \mathcal{Y}$ induces a homomorphism of local rings

$$y_0^{\#} : \mathcal{O}_{\mathcal{Y},\bar{y}_0} \to \mathcal{O}_{(v)}$$

where $\mathcal{O}_{(v)}$ is the localization of $\mathcal{O}$ at the ideal (v).

Since $\mathcal{O}_{\mathcal{Y},\bar{y}_0}$ is a regular local ring and $\mathcal{O}_{(v)}$ is a discrete valuation ring, there exists a regular system of parameters $\{p, z_1, \cdots, z_m\}$ of $\mathcal{O}_{\mathcal{Y},\bar{y}_0}$ such that $\{z_1, \cdots, z_m\}$ generates $\mathrm{Ker}(y_0^{\#})$. Then the completion of $\mathcal{O}_{\mathcal{Y},\bar{y}_0}$ can be identified with $\mathcal{O}_v[[z_1, \cdots, z_m]]$. Also, $\mathcal{O}_{\mathcal{Y},\bar{y}_0}$ can be identified as a subring of $\mathcal{O}_{(v)}[[z_1, \cdots, z_m]]$.

Suppose locally $A_i^j = \sum_k \Gamma_j^{ik} dz_k$ with

$$\Gamma_j^{ik} \in \mathcal{O}_{\mathcal{Y},\bar{y}_0} \subseteq \mathcal{O}_{(v)}[[z_1, \cdots, z_m]] \subseteq K[[z_1, \cdots, z_m]]$$

Now we turn back to the differential equations

$$d(f_i) = -\sum_j A_j^i f_j$$

It can be rewritten as

$$\frac{\partial f_i}{\partial z_k} = -\sum_j \Gamma_i^{jk} f_j$$

For any initial condition, we can obtain a formal solution $(f_1, f_2, \cdots, f_r) \in K[[z_1, \cdots, z_m]]^r$ and $f = \sum f_i v_i$. First, we discuss when it converges.

Lemma 2.3.1 (Picard-Lindelöf method)**.** *Any solution $\underline{f}$ is v-adic absolutely convergent in K_v on $\{(z_1, \cdots, z_m) : |z_i|_v < |p|_v^{\frac{1}{p-1}}\}$.*

Proof. Let $f_i = \sum a_I^{(i)} z_1^{i_1} \cdots z_m^{i_m} \triangleq \sum a_I^{(i)} z^I$. Through the differential equations

$$\sum_I a_I^{(i)} \cdot I_k z^{I-e_k} = -\sum_{j,I} \Gamma_i^{jk} a_I^{(i)} z^I$$

where I_k is the degree of z_k and $e_k = (0, \cdots, 1(k\text{th coordinate}), \cdots, 0)$. Then

$$a_I^{(i)} I_k + \sum_j \sum_{J+L=I-e_k} \Gamma_{i,J}^{jk} a_L^{(j)} = 0$$

where $\Gamma_{i,J}^{jk}$ is the coefficient of z^J in Γ_i^{jk}.

Recall that $\Gamma_{i,J}^{jk} \in \mathcal{O}_{(v)}$, its v-adic absolute value is not more than 1. Then

$$|I_k a_I^{(i)}|_v \leq \max |a_L^{(j)}|_v$$

since $|\cdot|_v$ is non-Archimedean.

Let $M_I = \max_{J \leq I,\ 1 \leq i \leq r} \{|a_J^{(i)}|_v\}$, then

$$M_I \leq \frac{M_{I-e_k}}{|I_k|_v}$$

Let $C(I)$ be the number $\prod |I_k!|_v$, then $C(I - e_k)|I_k|_v = C(I)$. By induction on n it follows that

$$M_I \leq \frac{M_0}{C(I)}$$

Since for any $n \in \mathbb{N}^+$, $|n!|_v > |p|_v^{\frac{n}{p-1}}$ (Legendre's formula), then

$$|f_i(z)|_v \leq \max |a_I^{(i)} z^I|_v \leq \frac{M_0}{|p|_v^{\frac{|I|}{p-1}}} \cdot \max |z_i|_v^{|I|}$$

■

By the choice of p, $p > 2$ and $|p|_v^{\frac{1}{p-1}} > |p|_v$. Also note that v is unramified, which implies that $|p|_v = |v|_v$. Then for $y_1 \equiv y_2 (\text{mod } v)$ for $y_1, y_2 \in \mathcal{Y}(\mathcal{O}_v)$, there is an isomorphism connecting the formal solutions on both sides

$$\nabla_{y_1,y_2} : H^q_{\mathrm{dR}}(X_{y_1}/K_v) \to H^q_{\mathrm{dR}}(X_{y_2}/K_v)$$

Theorem 2.3.2 (de Rham-crystalline comparison)**.** Let K be a number field. Assume that X is a smooth proper K_v-scheme and admits a good model, i.e., extends to a smooth proper $\mathcal{O}_{K_v}$-scheme $\mathcal{X}$, its generic fiber is X while its special fiber $X_{\mathbb{F}_v}$ is a $\mathbb{F}_v$-scheme. Then there exists a canonical K_v-linear isomorphism

$$\sigma : H^q_{\mathrm{dR}}(X/K_v) \cong H^q_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_{K_v}) \otimes_{\mathcal{O}_{K_v}} K_v \cong H^q_{\mathrm{cris}}(X_{\mathbb{F}_v}/W(\mathbb{F}_v))[1/p] \otimes_{W(\mathbb{F}_v)} K_v$$

which is compatible with cup-product.

Theorem 2.3.3 (Proposition 3.6.1 in [Ber06])**.** This isomorphism is compatible with the de Rham-crystalline comparison theorem 2.3.2. In particular, ∇_{y_1,y_2} preserves the Frobenius endomorphism φ at both sides and respects cup-product.

Chapter 3

Preliminary: Serre Torus and Average Weight

The main goal of this chapter is to prove 3.3.3, which limits the weights of local representations induced by global pure representations. This result will help us to deal with the non-semi-simple representations. To prove it, we need a result 3.3.1 on algebraic Hecke characters.

3.1 CM fields and friendly places

Let $\mathcal{C} \subseteq G_{\mathbb{Q}}$ be the conjugacy class of the complex conjugation, and let $H^+ = \langle \mathcal{C} \rangle$, the normal subgroup generated by $\mathcal{C}$. There is a nontrivial homomorphism $H^+ \to \{\pm 1\}$ sending h to $h(i)/i$ and we let H be its kernel.

We say an arbitrary number field $K \subseteq \bar{\mathbb{Q}}$ is totally real, if it is fixed by H^+. It is CM if it is fixed by H but not by H^+.

For an arbitrary number field K, let E and E^+ be, respectively, the subfields of K defined by fixed fields of $G_K \cdot H$ and $G_K \cdot H^+$. Then E^+ is the largest totally real subfield of K, and either $E^+ = E$ is totally real, or E is CM and is the largest CM subfield of K.

Definition 3.1.1. Let K be a number field.

- If K has a CM field, then let E be its maximal CM subfield and E^+ the maximal totally real subfield of E. In this case, we say that a place v of K is friendly if it is unramified over $\mathbb{Q}$, and it lies above a place of E^+ that is inert in E.
- If K has no CM subfield, any place v of K which is unramified over $\mathbb{Q}$ will be understood to be friendly.

3.2 Serre torus and locally algebraic representations

Now we need some results in algebraic groups. The main reference is [Ser97]. In this book, Serre reconstructed a type of Abelian p-adic representations of G_K for number field K firstly introduced by

Shimura, Tanitam and Weil. He also proved that any Abelian p-adic representation ρ is of this type if and only if it is "locally algebraic".

First we introduce the Serre torus and the construction of this type of representations.

Definition 3.2.1. Let $T = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m/K)$ be the algebraic group over $\mathbb{Q}$ obtained through Weil restriction. This algebraic group is indeed a torus of dimension $d = [K : \mathbb{Q}]$, that is, its base change over $\bar{\mathbb{Q}}$ is a d-times product of $\mathbb{G}_m/\bar{\mathbb{Q}}$. Moreover, every embedding $\sigma : K \to \bar{\mathbb{Q}}$ induces a homomorphism $K \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}$. Then by composing $T \to K$ we obtain a morphism $[\sigma] : T_{\bar{\mathbb{Q}}} \to \mathbb{G}_m/\bar{\mathbb{Q}}$. It is easy to check that the collection of all $[\sigma]$ gives the isomorphism

$$T_{\bar{\mathbb{Q}}} \to \prod_d \mathbb{G}_m/\bar{\mathbb{Q}}$$

Definition 3.2.2. Define the character group $X^*(T_{\bar{\mathbb{Q}}})$ to be the set of all characters, that is,

$$X^*(T) = \mathrm{Hom}_{\bar{\mathbb{Q}}}(T_{\bar{\mathbb{Q}}}, \mathbb{G}_m/\bar{\mathbb{Q}})$$

Then the set of $[\sigma]$ forms a basis of $X^*(T)$. Then, $G_{\mathbb{Q}}$ acts naturally on $X^*(T)$.

Proposition 3.2.3. The functor $T \mapsto X^*(T)$ sending a torus over k to its character group is an anti-equivalence of categories between the category of k-tori and the category of finite free $\mathbb{Z}$-modules equipped with a discrete G_K-action.

Proposition 3.2.4. Let E be a subgroup of $K^* = T(\mathbb{Q})$ and let $\bar{E}$ be the Zariski closure of E. It is an algebraic subgroup of T. Then we define T_E be the quotient $T/\bar{E}$. This is also a torus over $\mathbb{Q}$. Its character group is the subgroup of $X^*(T)$ defined by those characters which take value 1 on E.

Definition 3.2.5 ([Ser97], II 1.2)**.** Let A be a commutative algebraic group over a field k. Let $0 \to Y_1 \to Y_2 \to Y_3 \to 0$ be an exact sequence of commutative groups, with Y_3 finite. Let $Y_1 \to A(k)$ be a homomorphism. There exists a unique algebraic group B such that

(1) the following diagram is commutative

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & A(k) \arrow[d] \\ Y_2 \arrow[r] & B(k) \end{tikzcd}$$

(2) B is universal in the sense of (1).

At this case, we indeed have a commutative diagram for any field extension $k' \supseteq k$

$$\begin{tikzcd} 0 \arrow[r] & Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A(k') \arrow[r] & B(k') \arrow[r] & Y_3 \arrow[r] & 0 \end{tikzcd}$$

The algebraic group B is called an extension of the constant algebraic group Y_3 by A.

We now need some results from global class field theory. A good reference is [Mil11].

Let $\mathfrak{m} = (m_v)_{v\in S}$ be a modulus of support S, that is, $m_v \geq 1$ are integers and S is a subset of finite places of K. Set $U_\mathfrak{m} = (U_{\mathfrak{m},v})$ as

$$U_{\mathfrak{m},v} = \begin{cases} \text{connected component containing } 1, & v \text{ is infinite} \\ 1 + \pi_v^{m_v}\mathcal{O}_{K_v}, & v \in S \\ \mathcal{O}_{K_v}^*, & \text{otherwise} \end{cases}$$

Then it is a subgroup of Idèle $\mathbb{I}$. Let $E_\mathfrak{m} = K^* \cap U_\mathfrak{m}$, that is, $a \in K^*$ such that $a \in U_{\mathfrak{m},v}$ for all v. Denote by $C = C_K$ the Idèle class group $\mathbb{I}/K^*$, here we treat K^* as a subset of $\mathbb{I}$ through the inclusion $a \mapsto (a, a, \cdots)$. Then we have an exact sequence

$$1 \to K^*/E_\mathfrak{m} \to \mathbb{I}_\mathfrak{m} = \mathbb{I}/U_\mathfrak{m} \to C_\mathfrak{m} = C/K^*U_\mathfrak{m} \to 1$$

Global class field theory tells us that there is an isomorphism

$$C/D \cong \lim_{\leftarrow} C_\mathfrak{m} \cong G^{\mathrm{ab}}$$

where D is the connected component containing 1.

Note that $E_\mathfrak{m}$ defines a subgroup of $T(\mathbb{Q}) \cong K^*$. Thus, we can define $T_\mathfrak{m} = T/\overline{E_\mathfrak{m}}$ and we have an algebraic extension $S_\mathfrak{m}$ such that the diagram

$$\begin{tikzcd}
1 \arrow[r] & K^*/E_\mathfrak{m} \arrow[r] \arrow[d] & \mathbb{I}_\mathfrak{m} \arrow[r] \arrow[d] & C_\mathfrak{m} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & T_\mathfrak{m}(\mathbb{Q}) \arrow[r] & S_\mathfrak{m}(\mathbb{Q}) \arrow[r] & C_\mathfrak{m} \arrow[r] & 1
\end{tikzcd}$$

is commutative with the universal property.

Definition 3.2.6. Taking projective limit, the torus $\mathfrak{S} = \lim_{\leftarrow} T_\mathfrak{m}$ is called the Serre torus. A quick observation is that $\mathfrak{S}$ is the quotient of T by the Zariski closure of $\mathcal{O}_K^*$.

Set $\pi : T \to S_\mathfrak{m}$ to be the natural morphism between algebraic groups. Then π defines a homomorphism

$$\pi_p : T(\mathbb{Q}_p) \to S_\mathfrak{m}(\mathbb{Q}_p)$$

Note that $T(\mathbb{Q}_p) \cong \prod_{v|p} K_v^*$ is a direct factor of $\mathbb{I}$. Then there is a continuous homomorphism

$$\alpha_p : \mathbb{I} \to S_\mathfrak{m}(\mathbb{Q}_p)$$

The commutativity of the above diagram gives us that β and α_p coincide on K^*.

$$\begin{tikzcd}
 & & \mathbb{I} \arrow[d] \arrow[dd, bend right] & & \\
1 \arrow[r] & K^*/E_\mathfrak{m} \arrow[r, "\beta"] \arrow[d] & \mathbb{I}_\mathfrak{m} \arrow[r] \arrow[d] & C_\mathfrak{m} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & T/\bar{E}_\mathfrak{m}(\mathbb{Q}_p) \arrow[r] & S_\mathfrak{m}(\mathbb{Q}_p) \arrow[r] & C_\mathfrak{m} \arrow[r] & 1 \\
 & T(\mathbb{Q}_p) = \prod K_v^* \arrow[u] & & & \\
 & \mathbb{I} \arrow[u] \arrow[uur, "\alpha_p"'] & & &
\end{tikzcd}$$

Define $\epsilon_p = \beta \cdot \alpha_p^{-1}$. This defines a map $\mathbb{I} \to S_\mathfrak{m}(\mathbb{Q}_p)$. This actually induces a map $G^{\mathrm{ab}} \to S_\mathfrak{m}(\mathbb{Q}_p)$ since $S_\mathfrak{m}(\mathbb{Q}_p)$ is totally disconnected. Then for any character $S_\mathfrak{m} \to \mathbb{G}_m$, we can obtain a Hecke representation $\mathbb{I} \to G_K^{\mathrm{ab}} \xrightarrow{\epsilon_p} S_\mathfrak{m}(\mathbb{Q}_p) \to \mathbb{G}_m(\mathbb{Q}_p) = \mathbb{Q}_p^*$.

Now we give an explicit description of $X^*(\mathfrak{S})$.

We write $X^*(T)$ additively and let $Y(T) = X^*(T) \otimes_\mathbb{Z} \mathbb{Q}$. This is generated by $[\sigma]$ over $\mathbb{Q}$. We define

$$Y^0 = Y^G,\ Y^- = \{y : cy = -y,\ \forall c \in \mathcal{C}\}$$

and Y^+ is a G-invariant supplement to $Y^0 \oplus Y^-$ in Y.

Recall that $g\left(\sum_{\sigma: K \to \bar{\mathbb{Q}}} a_\sigma[\sigma]\right) = \sum a_\sigma[g\sigma]$. Then

$$Y^0 = \{y : y = a\left(\sum[\sigma]\right),\ a \in \mathbb{Q}\}$$

and its supplement

$$Y^- \oplus Y^+ = \{y = \sum a_\sigma[\sigma] : \sum a_\sigma = 0\}$$

Then every element $\chi \in X^*(T)$ can be written as

$$a\sum[\sigma] + \sum b_\sigma[\sigma]$$

where $\sum b_\sigma = 0$, and $a + b_\sigma \in \mathbb{Z}$.

In particular, Y^- can be described as

$$Y^- = \{y = \sum a_\sigma[\sigma] : \sum a_\sigma = 0,\ a_{c\sigma} = -a_\sigma\}$$

Proposition 3.2.7 ([Ser97] II 3.1 page II-30)**.** We have $X^*(T_\mathfrak{m}) \otimes_\mathbb{Z} \mathbb{Q} = Y^0 \oplus Y^-$.

Proposition 3.2.8. Define the norm map $\mathrm{Norm} : X^*(T_E) \to X^*(T_K)$ sending $[\sigma']$ for $\sigma' : E \hookrightarrow \bar{\mathbb{Q}}$ to the sum all $[\sigma]$ such that $\sigma : K \hookrightarrow \bar{\mathbb{Q}}$ restricts to σ'.

Then the norm map defines an isomorphism of Y_E^- onto Y_K^-.

Proof. The injectivity is obvious.

Conversely, if $\chi = \sum b_\sigma[\sigma] \in Y_K^-$, then $b_{h\sigma} = b_\sigma$, where $h = c_1 \cdots c_{2n} \in H$ and $c_i \in \mathcal{C}$. This shows that b_σ depends on b_{σ_E}, which means $\chi \in \mathrm{Im}(\mathrm{Norm})$. ∎

Remark 3.2.9. *The Norm map induces a morphism between algebraic groups* $\mathrm{Norm}_{K/E} : T_\mathfrak{m}^{(K)} \to T_\mathfrak{m}^{(E)}$ *by 3.2.3. Further, there is a commutative diagram*

$$\begin{tikzcd}
\mathbb{I}_K \arrow[r] \arrow[d, "\mathrm{Norm}_{K/E}"'] & K^*/E_\mathfrak{m} \arrow[r] \arrow[d, "\mathrm{Norm}_{K/E}"'] & T_\mathfrak{m}^{(K)}(\mathbb{Q}_p) \arrow[d, "\mathrm{Norm}_{K/E}"'] \\
\mathbb{I}_E \arrow[r] & E^*/E_\mathfrak{m} \arrow[r] & T_\mathfrak{m}^{(E)}(\mathbb{Q}_p)
\end{tikzcd}$$

Proposition 3.2.10 ([Ser97] II 3.3 page II-34)**.** The tori $T_\mathfrak{m}$ attached to K and E are isogenous with each other.

Now we turn to the discussion on locally algebraic characters.

Consider a continuous character $\eta : G_K \to \mathbb{Q}_p^*$ unramified outside S for a finite set S. We can associate η with a Hecke character. Indeed, note that η actually defines over G_K^{ab}. Then by global class field theory, η corresponds to a continuous homomorphism $C_K \to \mathbb{Q}_p^*$, where C_K is the Idèle class group. By restricting to the p-part, we have a homomorphism

$$\eta_p : (K \otimes \mathbb{Q}_p)^* \cong \prod_{v|p} K_v^* \to \mathbb{Q}_p^*$$

We will write $\eta_v = \eta|_{K_v^*} : K_v^* \to \mathbb{Q}_p^*$ as the map induced by η_p. Fix an inclusion $K \hookrightarrow K_v$, it is easy to check that η_v is equal to the composite $K_v^* \to G_{K_v}^{\mathrm{ab}} \hookrightarrow G_K^{\mathrm{ab}} \xrightarrow{\eta} \mathbb{Q}_p^*$, where the first map is the local Artin map.

Remark 3.2.11. *We say that the Hecke character $\eta : G_K \to \mathbb{Q}_p^*$ is unramified at a place v if η_v is trivial on $\mathcal{O}_{K_v}^*$. Note that this definition agrees with the unramification of representations, since the inverse image of I_v in K_v^* is exactly $\mathcal{O}_K^*$.*

For any continuous homomorphism $\mathbb{I} \to \mathbb{Q}_p^*$, where $\mathbb{I}$ is the Idèle group, it must send all but finitely many unit groups $\mathcal{O}_{K_v}^*$ to 1. See [Ser97] III. 2.2 page III-10 for the details. Hence, η is automatically unramified at almost all places.

Definition 3.2.12. We say that η_p is locally algebraic if there exists a neighborhood $U \subseteq \prod_{v|p} K_v^*$ of 1 and a morphism between algebraic groups $r : T \to \mathbb{G}_m$ such that $\eta_p(x) = r(x^{-1})$ for $x \in U \subseteq T(\mathbb{Q}_p) = \prod_{v|p} K_v^*$. We call $r_{\mathbb{Q}_p} : T_{\mathbb{Q}_p} \to \mathbb{G}_{m,\mathbb{Q}_p}$ the algebraic part of η_p.

Theorem 3.2.13 ([Ser97] III 2.2 proposition 2)**.** If η is locally algebraic, then it has a modulus of definition in the sense of [Ser97] III 2.2 page III-10.

Theorem 3.2.14 ([Ser97] III.2.3 theorem 2)**.** If η is locally algebraic and rational with modulus of definition $\mathfrak{m}$, then there exists a 1-dimensional $\mathbb{Q}$-vector space V and a morphism $\varphi : S_{\mathfrak{m}} \to \mathbb{G}_m$ of $\mathbb{Q}$-algebraic groups such that η is equal to the composite

$$\eta' : G_K^{\mathrm{ab}} \xrightarrow{\epsilon_p} S_{\mathfrak{m}}(\mathbb{Q}_p) \xrightarrow{\varphi_p} \mathbb{G}_m(\mathbb{Q}_p)$$

where φ_p is defined as the $\mathbb{Q}_p$-points of φ. Further, let $r : T_{\mathbb{Q}_p} \to \mathbb{G}_{m,\mathbb{Q}_p}$ be the algebraic part of η, then r factors through $r_{\mathfrak{m}} : T_{\mathfrak{m},\mathbb{Q}_p} \to \mathbb{G}_{m,\mathbb{Q}_p}$ and we have the composite

$$r_{\mathfrak{m}} : T_{\mathfrak{m},\mathbb{Q}_p} \to S_{\mathfrak{m},\mathbb{Q}_p} \xrightarrow{\varphi_{\mathbb{Q}_p}} \mathbb{G}_{m,\mathbb{Q}_p}$$

Also, note that $\eta_v(a_v) = \eta(a) = \varphi_p(\epsilon_p(a)) = \varphi_p(\alpha_p^{-1}(a)\beta(a)) = s(a_v^{-1})\varphi_p(\beta(a))$ for any $a_v \in K_v^*$ and $a = (\cdots, 1, a_v, 1, \cdots) \in \mathbb{I}$.

$$\begin{tikzcd}
 & & & T_{\mathfrak{m}}(\mathbb{Q}_p) \arrow[r] \arrow[rr, bend left, "r_{\mathfrak{m}}\text{, the algebraic part}"] & S_{\mathfrak{m}}(\mathbb{Q}_p) \arrow[r, "\varphi_p"] \arrow[dd, equal] & \mathbb{G}_m(\mathbb{Q}_p) \\
1 \in U \arrow[r, hook] & \prod_{v|p} K_v^* \arrow[r, hook] & \mathbb{I} \arrow[ur] \arrow[urr, bend right, "\alpha_p"'] \arrow[dr] \arrow[drr, bend right, "\epsilon_p"'] & & & \\
K_v^* \arrow[ur, hook] & & & G_K^{\mathrm{ab}} \arrow[r] \arrow[rr, bend right, "\eta=\eta'"'] & S_{\mathfrak{m}}(\mathbb{Q}_p) \arrow[r, "\varphi_p"] & \mathbb{G}_m(\mathbb{Q}_p)
\end{tikzcd}$$

The following important result for locally algebraic representation is given by Tate.

Theorem 3.2.15 ([Ser97] III A.7 theorem 3)**.** The character η_p is locally algebraic if and only if it is Hodge-Tate when restricting to any K_v for $v|p$.

3.3 Motivic weight and Hodge-Tate weights

Lemma 3.3.1. *Let v be any friendly place of K, lying over $p \in \mathbb{Q}$. Given a continuous character $\eta : G_K \to \mathbb{Q}_p^*$ which is unramified outside a finite set of places, pure of weight w (i.e., $|\eta(\mathrm{Frob})|$ is a Weil number of weight w), and locally algebraic at p. Let $s = \eta|_{K_v}^*$. Then*

$$s^2 = \chi \cdot \mathrm{Norm}_{K_v/\mathbb{Q}_p}^w$$

where χ has finite order. In particular, w is even and the Hodge-Tate weight of η at the place v equals to $w/2$.

Proof. We first check that, if s^2 has the form $\chi \cdot \mathrm{Norm}_{K_v/\mathbb{Q}_p}^{w'}$, then $w' = w$. Note that we can choose $\lambda \in U$ such that it locally can be represented as $(\cdots, 1, \pi_v^m, 1, \cdots)$ in $\mathbb{I}/K^*$. Since π_v corresponds to the Frob_v in Artin map, we find that

$$|s^2(\lambda_v)| = |\eta_v^2(\lambda_v)| = |\eta^2(\lambda)| = q^{mw'}$$

where $q = \#\mathcal{F}_v$. By raising to a finite power, we can obtain that $w' = w$.

Now we prove the final statement. Suppose that $1 + \mathfrak{m}_v^N \subseteq U$. For any $a \in \mathcal{O}_{K_v}^*$, we can write a into $a_0 + m$ for $a_0 \in \zeta^{q-1}$ and $m \in \mathfrak{m}$. Then $a^{(q-1)q^N} \in U$. However, note that

$$(\eta_v^2 \chi_0^{-w})|_U = (s^2 \chi_0^{-w})|_U = \chi|_U, \quad \chi_0 \text{ is the } p\text{-adic cyclotomic character}$$

has finite image, $(\eta_v^2 \chi_0^{-w})(\mathcal{O}_{K_v}^*)^{(q-1)q^N}$ has finite image. Then $(\eta^2 \chi_0^{-w})(I_v) = (\eta_v^2 \chi_0^{-w})(\mathcal{O}_{K_v}^*)$ has finite image in $\mathbb{Q}_p^*$. Then η_v has Hodge-weight $w/2$ (see, for example, [Hon] III.1.1.8).

Now it remains to show that s has the form $\chi \cdot \mathrm{Norm}_{K_v/\mathbb{Q}_p}^{w'}$ for some $w' \in \mathbb{Q}$.

If K has no CM subfield, we have Y^- is empty, then 3.2.7 tells us that $X^*(T_\mathfrak{m})$ has dimension 1. Therefore, the character $T_\mathfrak{m} \to S_\mathfrak{m} \xrightarrow{\varphi} \mathbb{G}_m$ could be written as Norm^w for $w \in \mathbb{Q}$, where Norm is defined in 3.2.8.

Now we assume that K has a maximal CM subfield E. Then the norm map $T_\mathfrak{m}^{(K)} \to T_\mathfrak{m}^{(E)}$ is an isogeny. In other word, a suitable power s^k factors through the norm map for $k \in \mathbb{Q}$. Therefore, it suffices to prove the lemma for $K = E$. And we choose a place $v \in E$ below v. Then v is inert over E^+.

Denote by $\mathcal{K}$ the kernel of the norm map $\mathrm{Norm}_{E/E^+} : T_\mathfrak{m}^{(E)} \to T_\mathfrak{m}^{(E^+)}$. Let $x \mapsto \bar{x}$ be the complex conjugation. The rule $c : x \mapsto x/\bar{x}$ defines a map $(E \times_\mathbb{Q} R)^* \to (E \times_\mathbb{Q} R)^*$ for any $\mathbb{Q}$-algebra R, and then a morphism of algebraic groups $\theta : T_\mathfrak{m}^{(E)} \to T_\mathfrak{m}^{(E)}$. Note that the image of c is exactly the elements with norm 1, i.e., θ is actually defined as $T_\mathfrak{m}^{(E)} \to \mathcal{K}$.

Therefore, by computing the dimensions and the kernel, we have an isogeny

$$\mathrm{Nm} \times \theta : T_\mathfrak{m} \to \mathbb{G}_m \times \mathcal{K}$$

Similarly, raising s to a suitable power we can suppose that it factors through the right hand. By twisting it with a power of cyclotomic character, we can arrange that it is trivial on $\mathbb{G}_m$. Then we only need to check the case that s factors through θ. Denote by s' the map

$$s' : E_v^* \to \prod_{v|p} E_v^* \to T_{\mathfrak{m}}(\mathbb{Q}_p) \to \mathcal{K}(\mathbb{Q}_p) = \mathrm{Ker}(\prod_{w|p} E_w^* \xrightarrow{\text{Norm}} \prod_{w|p} E_w^{+*})$$

such that s factors through s'. Then there exists $w = v$ such that $s'(a) \in E_v^*$ and $s'(a)s'(\bar{a}) = 1$. The image of it is contained in a $\mathbb{Q}_p$-anisotropic subtorus. Thus, any character of $\mathcal{K}$ is trivial over E_v^*. ■

Now we can link the global and local weights. Define the weight of a filtration $F^\bullet V \subseteq V$ to be

$$\mathrm{weight}_F(V) = \frac{\sum_{s\geq 0} s \dim \mathrm{gr}^s(V)}{\dim V}$$

where $\mathrm{gr}^s(V) = F^s(V)/F^{s+1}(V)$.

Proposition 3.3.2. Let K be a number field and v a friendly place. Let V be a Galois representation of G_K on a $\mathbb{Q}_p$-vector space which is crystalline at all primes above p, and pure of weight w.

Let V^{dR} be the image of $V|_{G_{K_v}}$ under D_{cris}. It has a K_v-filtered vector space structure.

Then the weight of the Hodge filtration on V^{dR} equals $w/2$.

Proof. Apply the above lemma to $\det(V)$. ■

Lemma 3.3.3. *Let K be a number field, and $L \supseteq K$ a finite extension. Let $\rho : G_L \to \mathrm{GL}_n(\mathbb{Q}_p)$ be a representation of G_L that is crystalline at all primes above p, and pure of weight w; let $a_u(\rho)$ be the weight of the associated Hodge filtration at each such prime u. Then for any friendly prime v of K above p,*

$$\sum_{u|v} [L_u : K_v] a_u(\rho) = [L : K]\frac{w}{2}$$

Proof. Apply above proposition to $\mathrm{Ind}_{G_L}^{G_K}\rho$ and v. Since we have

$$(\mathrm{Ind}_{G_L}^{G_K}\rho \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_{K_v}} \cong \bigoplus_{w|v} (\rho \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_{L_u}}$$

as filtered vector spaces, the result follows. ■

Remark 3.3.4. *In [Cor], Brian Conrad gave a easier proof of 3.3.3 for the case $K = \mathbb{Q}$.*

Chapter 4

Warm-up: The S-Unit Equation

In this chapter we will present how the method discussed in section 1.1.3 works. We will introduce a proof of the following famous result.

Theorem 4.0.1 (S-unit equation)**.** The set

$$U = \{t \in \mathcal{O}^* | 1 - t \in \mathcal{O}^*\}$$

is finite, where $\mathcal{O} = \mathcal{O}_S$ as usual.

If you hope to read the proof of the Mordell conjecture, you can skip this chapter.

Remark 4.0.2. *The solutions U of the S-unit equation and the set $K \backslash \{0, 1\}$ can both be equipped with an equivalence as*

$$t \sim (1 - t) \sim \frac{1}{t}$$

The orbit of any t is

$$\left\{t, 1 - t, \frac{1}{t}, \frac{1}{1-t}, \frac{t}{t-1}, \frac{t-1}{t}\right\}$$

There are the following two correspondences:

- *the set of elliptic curves with full 2-torsion points over K up to the isomorphism over $\overline{K}$ and elements $\lambda \in K \backslash \{0, 1\}$ (or points $P \in \mathbb{P}^1(K) \backslash \{0, 1, \infty\}$) modulo the orbits defined above.*
- *The set of elliptic curves with full 2-torsion points over K up to the isomorphism over $\overline{K}$ with potentially good reduction outside S and U modulo the orbits defined above.*

The first correspondence can be treated as the following sense: every elliptic curve E is isomorphic to the elliptic curve E_λ given by the Legendre form

$$E_\lambda : y^2 = x(x - 1)(x - \lambda)$$

over $\overline{K}$. The assumption E has full 2-torsion points implies $\lambda \in K$. III.1.7 in [Sil09] tells us why this defines an equivalence.

For the second one, the Legendre form has potentially good reduction outside S if and only if the j-invariant

$$j(E_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

is in $\mathcal{O} = \mathcal{O}_S$. Then $\lambda \in U$.

The whole proof of the finiteness of U can be divided into three steps.

Proof. 1. First we do some reductions. Note that if we enlarge S to S' or enlarge K to a finite extension K', then $U \subseteq U'$. Hence, we may assume that S contains all primes above 2 and K contains the 8-th roots of unity. Let $m = 2^n$ be the largest power of 2 such that $\mu_m \subseteq K$. By the assumption, $m \geq 8$.

Let $U_1 = \{t \in \mathcal{O}^* | 1 - t \in \mathcal{O}^*, t \notin (K^*)^2\}$. If $t \in U$, assume that n is the maximal number such that t can be written as s^m for $s \in \mathcal{O}^*$. Suppose $(1 - s^m)b = 1$, then $1 - s$ has an inverse element $(1 + s + \cdots + s^{m-1})b$. Then $s \in U_1$ and $t \in U_1^m$. If $t = (t')^m$ for some $t' \in (\mathcal{O}^*)^2$, then $t = (t' \cdot \mu_m)^m$, and $t' \cdot \mu_m \notin (\mathcal{O}^*)^2$. Hence,

$$U = U_1 \cup U_1^2 \cup \cdots \cup U_1^m$$

Then we only need to show that the set U_1 is finite.

For any $t \in U_1$, its order in $K^*/(K^*)^m$ is m. By the Kummer theory, this element corresponds to an Abelian extension $K \subseteq K(t^{1/m})$ with Galois group $\mathbb{Z}/m\mathbb{Z}$. For any $\mathfrak{p} \notin S$, $\mathfrak{p}$ is not above 2, and $|t|_{\mathfrak{p}} = 1$, then $\mathfrak{p}$ is unramified in $K(t^{1/m})/K$. By a theorem from Hermit, there are only finitely many isomorphism classes as the form $K(t^{1/m})$. Then it suffices to prove the following set

$$U_{1,L} = \{t \in U_1 | K(t^{1/m}) \cong L\}$$

is finite for a fixed field extension L/K.

We may write $L = K(t_0^{1/m}) \cong K[x]/(x^m - t_0)$ for some t_0.

Choose a place v of K such that

(1) the class of Frob_v generates $\mathrm{Gal}(L/K)$.

(2) The rational prime p below v is unramified in K.

(3) No prime of S is above p.

Let $L_v = L \otimes K_v = L_v[x]/(x^m - t)$. Note that the first condition implies that v is inert in L/K, i.e., v remains prime in L. Then t is not a square in K_v, for otherwise L_v is not a field.

$$\begin{array}{ccc} L & & L_v \\ \uparrow & & \uparrow \\ K & & K_v \\ \uparrow & & \uparrow \\ \mathbb{Q} & & \mathbb{Q}_p \end{array}$$

Fix $t_0 \in U_{1,L}$, through the maps

$$\mathcal{O}^* \to \mathcal{O}_v^* \to \mathbb{F}_v$$

it suffices to prove the set

$$U_{1,L,v} = \{t \in U_{1,L} | t \equiv t_0 (\mathrm{mod}\ v)\}$$

is finite.

2. Now we do some preparatory works.

Let $\mathcal{Y} = \mathbb{P}^1_{\mathcal{O}} \backslash \{0, 1, \infty\}$ and $\mathcal{Y}' = \mathbb{P}^1_{\mathcal{O}} \backslash \{0, \mu_m, \infty\}$. Let $\pi : \mathcal{Y}' \to \mathcal{Y}$ be the map $x \mapsto x^m$. Let $\mathcal{X} \to \mathcal{Y}'$ be the Legendre family, i.e., defined by $y^2 = x(x-1)(x-t)$, and the map sending a solution to t. The composite is $\pi : \mathcal{X} \to \mathcal{Y}$.

Denote by X, Y the generic fibers of $\mathcal{X}, \mathcal{Y}$ over K. These points correspond to those elements in $\mathcal{O}^*$. Then the fiber X_t of the map $X \to Y$ induced by π is the disjoint union of the curves $y^2 = x(x-1)(x-z)$ where $z^m = t$ for $t \in Y(K)$. For $t \in Y(K)$, we can associate a Galois representation ρ_t of G_K on $H^1_{\text{ét}}(\bar{X}_t, \mathbb{Q}_p) = \bigoplus_{z^m=t} H^1_{\text{ét}}(\bar{X}_z, \mathbb{Q}_p)$.

We temporarily assume the rightness of the following two lemmas, which we will prove later in the next section.

Lemma 4.0.3 (Big monodromy)**.** *Consider the family of curves over $\mathbb{C}\backslash\{0,1\}$ whose fiber over $t \in \mathbb{C}$ is the disjoint union of elliptic curves $y^2 = x(x-1)(x-z)$ for all m-th roots $z^m = t$. Then the action of monodromy*

$$\pi_1(\mathbb{C}\backslash\{0,1\}, t_0) \to \operatorname{Aut}\left(\bigoplus_{z^m=t_0} H^1_{\text{sing}}(E_z, \mathbb{Q})\right)$$

has Zariski closure containing $\prod_z \mathrm{SL}(H^1_{\text{sing}}(E_z, \mathbb{Q}))$.

Lemma 4.0.4. *Fix a number field L, and a rational prime p larger than 2 and unramified in L. There are only finitely many $z \in L$ such that z, $1-z$ are p-units but for which the Galois representation of G_L on the Tate module $T_p(E_z) \cong H^1_{\text{ét}}(\bar{E}_z, \mathbb{Q}_p)$ of the elliptic curve*

$$E_z : y^2 = x(x-1)(x-z)$$

fails to be simple.

Remark 4.0.5. *We have known a very general version of this lemma 4.0.4, as a part of Tate conjecture over number fields, which was proved by Faltings when he proved Mordell conjecture in his paper [Fal86].*

3. Now we finish the proof. We exactly follow the steps discussed in section 1.1.3. The S-unit condition implies that the representations ρ_z are unramified outside S. Hence we can use the results in section 1.1.3 safely. In this special case, there are something making the proof easier, for example, $w = v$ and $\kappa(z) \cong K(t^{1/m})$ for all $z \in \pi^{-1}(t)$.

Note that 4.0.4 tells us that we can use 1.1.5 conveniently. From 1.1.5, $\rho_t|_{G_{K(t^{1/m})}}$ lies in finite isomorphism classes. Hence, it is enough to prove the set

$$U_1' = \left\{ t \in U_{1,L,v} : \rho_t|_{G_{K(t^{1/m})}} \text{ is isomorphic to } \rho_{t_0}|_{G_{K(t^{1/m})}} \right\}$$

is finite. Also, note that this is a subset of

$$U_2' = \left\{ t \in U_{1,L,v} : \rho_t|_{G_{K_v(t^{1/m})}} \text{ is isomorphic to } \rho_{t_0}|_{G_{K_v(t^{1/m})}} \right\}$$

it suffices to show that U_2' is finite.

From p-adic Hodge theory, we know that the representation ρ_t corresponds to the filtered φ-module

$$(H^1_{\text{dR}}(X_{t,K_v}/K_v), \varphi_v, F^i)$$

the restriction $\rho_t|_{G_{K_v(t^{1/m})}}$ means we treat $H^1_{\mathrm{dR}}(X_{t,K_v}/K_v)$ as $K_v(t^{1/m})$-module, which we will describe more explicitly.

Note that in the fraction $X \to Y' \to Y$, X_t for $t \in Y$ there exists a morphism $X_t \to Y'_t$. Then X_t equips naturally with a $\mathrm{Spec}(K(t^{1/m}))$-scheme structure through the map $Y'_t \to \mathrm{Spec}(K(t^{1/m}))$. The de Rham cohomology does not depend on how we treat it — a $2m$-dimensional K-vector space or a 2-dimensional $K(t^{1/m})$-vector space.

As we illustrated in the last chapter, there is a parallel transportation for t and t_0 with $t \equiv t_0 (\mathrm{mod}\ v)$:

$$H^1_{\mathrm{dR}}(X_{t,K_v}/K_v) \cong H^1_{\mathrm{dR}}(X_{t_0,K_v}/K_v)$$

This is compatible with the isomorphism $K_v(t^{1/m}) \cong K_v(t_0^{1/m})$.

Note that the Hodge filtration of H^1_{dR} has only one nonzero term F^1. The image of $\{t : t \equiv t_0 (\mathrm{mod}\ v)\}$ under Φ_v is a $K_v(t_0^{1/m})$-line, which lies in $Z(\varphi_v)$. We can control it by applying 1.1.14. Therefore, $\dim \Phi(t) \leq \dim_{K_v(t_0^{1/m})} H^1_{\mathrm{dR}} = 4$.

Now, if we want to use 1.1.22, we only need to give a description of $\dim_{\mathbb{C}}(\Gamma \cdot h_0^{\mathbb{C}})$. Then we turn to the complex analytic variety $X_{t_0,\mathbb{C}}$. Note that it is a disjoint union of m elliptic curves C_i. Also, the de Rham cohomology H^1_{dR} is given by the direct sum of each part, i.e., $H^1_{\mathrm{dR}}(X_{t_0,\mathbb{C}}/\mathbb{C}) = \bigoplus_{i=1}^m H^1_{\mathrm{dR}}(C_i/\mathbb{C})$. The Hodge filtration also splits with this decomposition. Then 4.0.3 tells us that $\Gamma \cdot h_0^{\mathbb{C}}$ is the full flag variety $\prod \mathbb{P}^1_{\mathbb{C}}$. In particular, the dimension is m, which is bigger than 4. Then the result follows from 1.1.22. ■

4.1 The completion of the proof

We first give a proof for 4.0.4.

Proof of 4.0.4. Fix z_0 with the required properties. Then by direct calculation we find that E_z has good reduction at all places v of L above p.

Similarly, by considering the class in $\mathbb{F}_v$, i.e., the image of z_0 under the composite $L \to L_v \to \mathbb{F}_v$, we only need to show that

$$V_L = \{z \in L : z \equiv z_0 (\mathrm{mod}\ v),\ \forall v|p,\ T_p(E_z) \text{ is not a simple representation}\}$$

is finite.

Note that $T_p(E_z)$ is reducible. Then there exists a sub-representation $W_z \subseteq T_p(E_z)$ with dimension 1. We already know that (for example, [vdGM07] 16.4) $T_p(E_z)$ is pure of weight 1. By the character theory of matrices, W_z is pure of weight 1.

Since the sub-representations of a crystalline representation is also crystalline, using 3.3.3 for $L/\mathbb{Q}$, there exists a place w of L above p such that $F^1(W_z^{\mathrm{dR}}) = W_z^{\mathrm{dR}}$, where $(W_z^{\mathrm{dR}}, \varphi, F^1 W_z^{\mathrm{dR}})$ denotes the filtered φ-submodule $D_{\mathrm{cris}}(W_z) \subseteq D_{\mathrm{cris}}(T_p(E_z))$. Also, we will use the notation W_z^{cris} to denote the corresponding K_0-isocrystal.

Note that $W_z^{\mathrm{dR}} \cap F^1(H^1_{\mathrm{dR}}(E_z/L_w)) = F^1(W_z^{\mathrm{dR}})$ has dimension 1, then $W_z^{\mathrm{dR}} = F^1(H^1_{\mathrm{dR}}(E_z/L_w)) = F^1(W_z^{\mathrm{dR}})$, thus, we can exactly control $F^1(H^1_{\mathrm{dR}}(E_z/L_w))$ using this sub-representation!

A famous result is that Hodge and Newton polygons of W_z^{dR} have the same end point ([BC09] Theorem 9.3.4). Then $t_H(W_z^{\mathrm{cris}}) = t_N(W_z^{\mathrm{cris}}) = 1$. Then $F^1(H^1_{\mathrm{dR}}(E_z/L_w))$ is exactly the slope-1 subspace for φ.

Consider the line

$$V_L \to \mathcal{F}_w, \quad z \mapsto \nabla(F^1(H^1_{\mathrm{dR}}(E_z/L_w))) \subseteq H^1_{\mathrm{dR}}(E_{z_0}/L_w)$$

in the flag variety. For $z \in V_L$, since ∇ is commutative with φ, the image is fixed, the line with eigenvalue 1 for φ. Then V_L maps to one point. Then by Torelli theorem for elliptic curves the preimage of one point is finite. ■

4.1.1 The proof of 4.0.3

We complete this chapter by giving a proof of 4.0.3. We will follow the method available in [Zav]. The main idea is to express the monodromy action through the usual Legendre family.

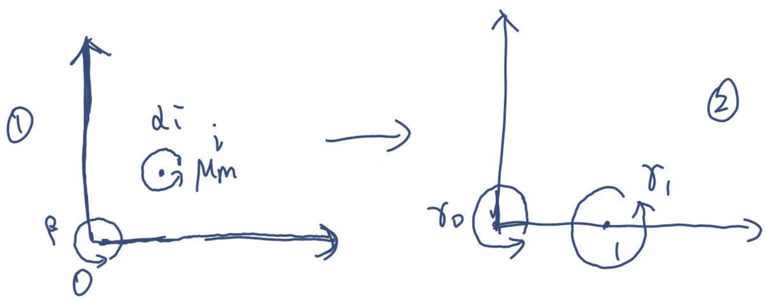
We denote by S the curve $\mathbb{A}^1_{\mathbb{C}}\backslash\{0,1\}$ (only in this section), and S' the curve $\mathbb{A}^1_{\mathbb{C}}\backslash\{0,\mu_m(\mathbb{C})\}$, here $\mu_m(\mathbb{C})$ represents all m-unit of roots. Then there is a natural morphism $h : S' \to S$ induced by the map $x \mapsto x^m$. Denote by $g : X \to S'$ the Legendre family over S', then we are interested in the composite

$$p : X \xrightarrow{g} S' \xrightarrow{h} S$$

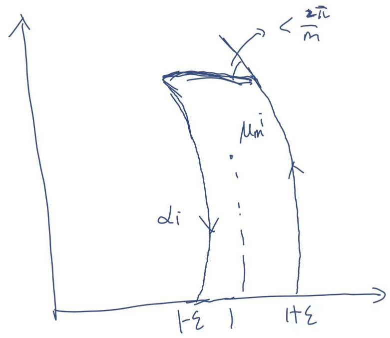
Denote by Γ and Γ' the fundamental groups $\pi_1(\mathbb{C}\backslash\{0,1\}, t)$ and $\pi_1(\mathbb{C}\backslash\{0,\mu_m(\mathbb{C})\})$, then the map $S' \to S$ induces a natural map $\varphi : \Gamma' \to \Gamma$. This map is obviously injective since $S' \to S$ is a cover. Further, we have the following lemma.

Lemma 4.1.1. *Denote by $\alpha_i \in \Gamma'$ the loop around μ_m^i oriented counterclockwise, and $\beta \in \Gamma'$ the loop around 0 oriented counterclockwise. Denote by γ_0, γ_1 the loops around $0, 1$ oriented counterclockwise separately. Then*

$$\varphi(\beta) = \gamma_0^m, \ \varphi(\alpha_i) = \gamma_0^{-i}\gamma_1\gamma_0^i$$



Proof. For β, it is well-known. For α_i, we use the following graph.



For the loop α_i, the rising part of α_i will map to i-copies of γ_0, the "horizontal" part closed to μ_m^i will contribute to γ_1, and the descending part contributes to γ_0^{-i}. ■

Theorem 4.1.2 ([Zav], Cor 3.6)**.** The representation of the standard Legendre family could be given by

$$r_0 \mapsto \begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix}, \quad r_1 \mapsto \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}$$

Lemma 4.1.3. *For any sheaf of Abelian group $\mathcal{F}$ on S' and any $i > 0$, all higher direct images $R^i h_* \mathcal{F}$ vanishes. In particular, $R^i p_*(\mathbb{Q}) = h_*(R^i g_*(\mathbb{Q}))$. Moreover, $R^1 p_* \mathbb{Q}$ is a $\mathbb{Q}$-local system of rank $2m$, and $R^1 g_* \mathbb{Q}$ is a $\mathbb{Q}$-local system of rank 2.*

Proof. Locally, the cover is trivial. Thus, h_* is exact. Then the following statement follows immediately.

For the second one, we consider a Leray-Serre spectral sequence

$$E_2^{ij} = R^i h_*(R^j g_* \mathbb{Q}) \Rightarrow R^{i+j} p_* \mathbb{Q}$$

Then $R^i p_* \mathbb{Q} \cong R^0 h_*(R^j g_* \mathbb{Q})$.

For the last part, note that f and g are both smooth and proper morphisms of finite type $\mathbb{C}$-schemes. Their analytifications are proper submersions. Then we use Ehresmann theorem and proper base change theorem so that $R^i f_*$ and $R^i g_*$ send local systems to local systems. Thus, we can compute the ranks in fibers. ■

Lemma 4.1.4. *Denote by $(V, \rho_{S'})$ the representation of Γ' induced by $g : X \to S'$ and the $\mathbb{Q}$-local system $R^1 g_* \mathbb{Q}$. Then $\rho_{S'}(\alpha_i) = e$ for any $i > 0$ and*

$$\rho_{S'}(\alpha_0) = \begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix} \quad \rho_{S'}(\beta) = \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}$$

Proof. Indeed, there is a commutative diagram

$$\begin{tikzcd} \Gamma' \arrow[r, hook, "j"] \arrow[dr, "\rho_{S'}"'] & \Gamma \arrow[d, "\rho_S"] \\ & \mathrm{GL}(H^1(E_z, \mathbb{Q})) \end{tikzcd}$$

where j is the map induced by the immersion $S' \to S$ (not φ). Obviously $j(\alpha_i) = e$ for $i > 0$, $j(\alpha_0) = \gamma_1$ and $j(\beta) = \gamma_0$. Then the results follow from 4.1.2. ■

Now we want to understand the representation $\mathrm{GL}(H^1(X_t,\mathbb{Q}))$, which is the push forward of the representation $(V,\rho_{S'})$ to S, by 2.1.11, it is isomorphic to $\mathrm{Ind}_{\Gamma'}^{\Gamma}(V,\rho_{S'})$ (here we use the map $\varphi:\Gamma'\to\Gamma$).

Then we make all the things into the representation theory.

Note that the quotient $\Gamma/\varphi(\Gamma')$ has representatives $\{1,\gamma_0,\cdots,\gamma_0^{m-1}\}$. By the definition of induced representations, $\mathrm{Ind}_{\Gamma'}^{\Gamma}(V,\rho)$ can be factored as

$$\mathrm{Ind}_{\Gamma'}^{\Gamma}(V,\rho)\cong\bigoplus_{i=0}^{m-1}\gamma_0^i\cdot V$$

where the action of Γ is defined in the following way: for any $v=\sum\gamma_0^i\cdot v_i$,

$$\rho_S(\gamma)\cdot v=\sum\gamma_0^{j(i)}\cdot(\rho_{S'}(h_i)v_i)$$

where $j(i)$ is a unique number such that there exists $h_i\in\Gamma'$ with a property $\gamma\gamma_0^i=\gamma_0^{j(i)}\varphi(h_i)$. By the way, we have the containment

$$\mathrm{GL}(H^1(X_t,\mathbb{Q}))=\mathrm{GL}\left(\bigoplus_{i=0}^{m-1}\gamma_0^i\cdot V\right)\supseteq\prod_{i=0}^{m-1}\mathrm{SL}_2(\gamma_0^iV)$$

Then we have the following computations.

For γ_1, note that $\gamma_1\gamma_0^i=\gamma_0^i\varphi(\alpha_i)$, then $j(i)=i$ and $h_i=\alpha_i$. Thus

$$\rho_S(\gamma_1)(\gamma_0^iv)=\gamma_0^i(\rho_{S'}(\alpha_i)(v))=(\text{by } 4.1.4)\begin{cases}\rho_0^iv, & i>0\\ \gamma_0^0\cdot\begin{pmatrix}1&\pm2\\0&1\end{pmatrix}\cdot v, & i=0\end{cases}$$

Then $\rho_S(\gamma_1)=\mathrm{diag}(u,1,\cdots,1)$, where $u\in\mathrm{SL}_2(\mathbb{C})$ a unipotent matrix.

For γ_0, note that $\gamma_0\gamma_0^i=\gamma_0^{i+1}\varphi(1)$, for $i<m-1$, then if $w\in\gamma_0^iV$, we have $\rho_S(\gamma_0)w\in\gamma_0^{i+1}V$. If $i=m-1$, then $\gamma_0\gamma_0^{m-1}=\varphi(\beta)$ and

$$\rho_S(\gamma_0)(\gamma_0^{m-1}v)=\gamma_0^0\cdot\begin{pmatrix}1&0\\\pm2&1\end{pmatrix}\cdot v$$

Then we can give a description of $\overline{\mathrm{Im}(\rho_S)}$. Note that $\rho_S(\gamma_0^m)$ and $\rho_S(\gamma_1)$ preserve the decomposition. Then the image contains all the forms $\begin{pmatrix}1&2n\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\2k&1\end{pmatrix}$ for $n,k\in\mathbb{Z}_+$ at each part. Then by taking the closure, $\overline{\mathrm{Im}(\rho_S)}$ contains $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\1&1\end{pmatrix}$, and then SL_2.

Chapter 5

Proof towards Mordell conjecture

In the remaining two chapters, we assume that $Y = C$ is a given projective curve over K with genus $g \geq 2$. We hope to use the results in section 1.1.3 to prove that $Y(K)$ is finite.

5.1 Some preparatory works

We first consider a special type of families over Y, we explained it in the example 1 before.

Definition 5.1.1. An Abelian-by-finite family over Y/K is a sequence of smooth proper morphisms $X \to Y' \xrightarrow{\pi} Y$, where π is finite étale and $X \to Y'$ is a polarized Abelian scheme over K.

A good model for such a family over $\mathcal{O} = \mathcal{O}_{K,S}$ is a family $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$ of smooth proper $\mathcal{O}$-schemes satisfying the same conditions as above and recovering $X \to Y' \to Y$ over K.

To deal with the monodromy requirement, we will consider families in the following condition.

Definition 5.1.2. Consider a complex point $y_0 \in Y(\mathbb{C})$, the fiber X_{y_0} is a disjoint union of finite fibers X_z, where $z \in Y'(\mathbb{C})$ such that $\pi(z) = y_0$. Consider the monodromy action on the first singular cohomology

$$H^1_{\text{sing}}(X_{y_0}, \mathbb{Q}) \cong \bigoplus_{\pi(z)=y_0} H^1_{\text{sing}}(X_z, \mathbb{Q})$$

The family is said to have full monodromy if the Zariski closure of the image of $\pi_1(Y_{\mathbb{C}}(\mathbb{C}), y_0)$ contains the product of symplectic groups

$$\overline{\text{Im}(\pi_1(Y_{\mathbb{C}}(\mathbb{C}), y_0))} \supseteq \prod_{\pi(z)=y_0} \text{Sp}(H^1_{\text{sing}}(X_z, \mathbb{Q}), \omega)$$

where the symplectic group is associated with the polarization ω on the singular cohomology of the fibers X_z.

Remark 5.1.3. *Fix an Abelian-by-finite family $X \to Y' \to Y$ over K which admits a good model over $\mathcal{O}$. Let $E_y = \Gamma(\mathcal{O}_{(Y')_y})$ be the global section of the fiber $(Y')_y$ for any $y \in Y(K)$. It is a finite étale K-scheme and has a decomposition $E_y = \prod_{z \in \pi^{-1}(y)} \kappa(z)$. The fiber X_y is a Abelian scheme over E_y with each fiber X_z a polarized Abelian variety.*

The first de Rham cohomology also has a corresponding decomposition

$$V_y \triangleq H^1_{\mathrm{dR}}(X_y/K) = H^1_{\mathrm{dR}}(X_y/E_y) = \bigoplus_{z\in\pi^{-1}(y)} H^1_{\mathrm{dR}}(X_z/\kappa(z))$$

The polarization gives an E_y-bilinear symplectic pairing

$$\theta_y : H^1_{\mathrm{dR}}(X_y/K) \times H^1_{\mathrm{dR}}(X_y/K) \to E_y$$

By tensoring K_v for a place $v \notin S$ we obtain a decomposition in v-adic version

$$V_{y,v} = V_y \otimes_K K_v = H^1_{\mathrm{dR}}(X_y/K_v) = H^1_{\mathrm{dR}}(X_y/E_{y,v}) = \bigoplus_{z\in\pi^{-1}(y),w|v} H^1_{\mathrm{dR}}(X_z/\kappa(z)_w)$$

where $E_{y,v} = E_y \otimes_K K_v = \prod_{z\in\pi^{-1}(y),w|v} \kappa(z)_w$ and w takes value in the set of the places of $\kappa(z)$.

Note that the G_K-action on the set $\pi^{-1}(y)(\bar{K})$ is unramified outside S. To control the degree $[\kappa(z)_w : K_v]$, it is equivalent to consider the Frob_v-action on z. Then the following definition will help us to reflect the sizes of $[\kappa(z)_w : K_v]$.

Definition 5.1.4. Let E be a finite G_K-set and v a place of K such that G_K-action on E is unramified at v. Define

$$\mathrm{size}_v(E) = \frac{\#\{x \in E : |\mathrm{Frob}_v\text{-orbit of } x| < 8\}}{\#E}$$

Lemma 5.1.5. *If $E \to E'$ is a G_K-invariant map of finite G_K-sets, and all fibers have the same cardinality, then $\mathrm{size}_v(E) \leq \mathrm{size}_v(E')$.*

Proof. It's obvious. ∎

5.2 A giant reduction

The advantage of taking Abelian-by-finite family is that we can use 1.1.22 to show the following result.

Proposition 5.2.1. Let Y be a smooth projective curve of genus $g \geq 2$ over K. Let $X \to Y' \xrightarrow{\pi} Y$ be an Abelian-by-finite family over Y, with full monodromy. Let d be the relative dimension $X \to Y'$. Assume the family admits a good model over the ring $\mathcal{O} = \mathcal{O}_{K,S}$ for some finite set of places S and choose a friendly place $v \notin S$ of K. Then the set

$$Y(K)^* = \left\{ y \in Y(K) | \mathrm{size}_v(\pi^{-1}(y)(\overline{K})) < \frac{1}{d+1} \right\}$$

is finite.

Under 5.2.1, all we need to do is to find an Abelian-by-finite family over Y, which doesn't have z with big "size". Fortunately, this is true. See the proposition and the proof below.

Proposition 5.2.2. For each prime $q \geq 3$, there exists a specific Abelian-by-finite family $X_q \to Y'_q \xrightarrow{\pi} Y$, which we will introduce explicitly in the subsequent chapter. It has the following properties:

(i) It has full monodromy.

(ii) The relative dimension d_q of $X_q \to Y'_q$ is given by $d_q = (q-1)(g-\frac{1}{2})$.

(iii) For each $y_0 \in Y(K)$ there is a G_K-equivalent identification of $\pi^{-1}(y_0)$ with the conjugacy classes of surjections $\pi_1^{\mathrm{geo}}(Y-y_0,*) \to \mathrm{Aff}(q)$ that are nontrivial on a loop around y_0, where

$$\mathrm{Aff}(q) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_q^*, b \in \mathbb{F}_q \right\}$$

and π_1^{geo} the geometric étale fundamental group.

We will introduce the construction in the next chapter. We currently admit this fact.

Remark 5.2.3. *The group $\pi_1^{\mathrm{geo}}(Y-y_0,*)$ can be identified as the profinite completion of a free non-commutative group generated by $\{a_1, b_1, \cdots, a_g, b_g\}$ and the loop $\prod[a_i,b_i] = \prod a_i b_i a_i^{-1} b_i^{-1}$. Hence,*

$$\pi_1^{\mathrm{geo}}(Y-y_0,*)^{\mathrm{ab}} \cong \pi_1^{\mathrm{geo}}(Y,*)^{\mathrm{ab}}$$

Now we can prove the Mordell conjecture.

Proof for Mordell conjecture. First, if the conjecture holds for a finite extension K', then it holds for K. Hence, we may assume that $K/\mathbb{Q}$ is a Galois extension.

Note that the crucial point is to choose some prime q and a friendly place v of K, such that

$$Y(K)^* = Y(K)$$

that is,

$$\mathrm{size}_v(\pi^{-1}(y)(\overline{K})) < \frac{1}{d_q+1}, \ \forall y \in Y(K)$$

We want to choose a prime number q, such that

(i) $q \equiv 3 (\mathrm{mod}\ 4)$ and $q \equiv 2 (\mathrm{mod}\ \ell)$ for any odd prime $\ell | \mathrm{disc}(K)$ or $\ell \leq 8[K:\mathbb{Q}]$.

(ii) K is linearly disjoint from $\mathbb{Q}(\zeta_{q-1})$ over $\mathbb{Q}$.

(iii) $\dfrac{7\cdot 2^{g+1}}{(q-1)^g} < \dfrac{1}{(g-\frac{1}{2})(q-1)+1}$.

By Dirichlet theorem, we can choose q satisfying the (i) and (iii). If there exists p ramifies both in $\mathbb{Q}(\zeta_{q-1}) = \mathbb{Q}(\zeta_{\frac{q-1}{2}})$ and K, then $p | \frac{q-1}{2}$ and $p | \mathrm{disc}(K)$. If $p = 2$ then $4 | q-1$, if $2 \nmid p$ then $q \equiv 1 (\mathrm{mod}\ p)$, they both contradicts with (i). Then (ii) follows and then

$$\mathrm{Gal}(K(\zeta_{q-1})/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_{q-1})/\mathbb{Q})$$

is an isomorphism.

Then we obtain a corresponding Abelian-by-finite family $X_q \to Y'_q \xrightarrow{\pi} Y$ and we choose S such that the family can extend to a good model (through [Liu06] Prop 10.1.21 (i) and (ii)).

Next we choose $v \notin S$ such that

1. v is friendly.

2. $(q_v, q-1) = 1$, where q_v is the cardinality of the residue field at v.

3. For any odd prime factor r of $q-1$, the class of q_v in $(\mathbb{Z}/r\mathbb{Z})^*$ has order at least 8.

If K has no CM subfield, let $\Sigma = \mathrm{Gal}(K/\mathbb{Q})$. If K has a CM subfield, let E be the maximal CM subfield of K and K^+ the maximal totally real subfield, let $\Sigma \subseteq \mathrm{Gal}(K/E^+)$ be the set of elements which restrictions to $\mathrm{Gal}(E^+/E)$ generates $\mathrm{Gal}(E^+/E)$.

Choose $a \in (\mathbb{Z}/(q-1)\mathbb{Z})^*$ such that a is a primitive root in $(\mathbb{Z}/r\mathbb{Z})^*$ for any prime factor $r|q-1$ (that is, the minimal number n satisfying $a^n \equiv 1(\mathrm{mod}\ r)$ is $r-1$).

By the Chebotarev theorem, there is a place $\mathfrak{p}$ of $K(\zeta_{q-1})$ such that $\mathrm{Frob}_{\mathfrak{p}} \in \Sigma \times \{a\} \subseteq \mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_{q-1})/\mathbb{Q})$. Let $p \in \mathbb{Q}$ be the prime below $\mathfrak{p}$. Let $v \in K$ be the place below $\mathfrak{p}$, then the residue field at v has cardinality p^i for some $i \leq [K:\mathbb{Q}]$.

Note that $p \equiv a(\mathrm{mod}\ q-1)$, then v satisfies condition 2. For any odd prime factor $r|q-1$, the order of p^i in $(\mathbb{Z}/r\mathbb{Z})^*$ is at least $\left[\dfrac{r-1}{i}\right]$, which is obviously not less than 8 by the condition (i).

Also, by the choice of Σ, the place in E^+ below $\mathfrak{p}$ remains prime and then is inert in E^+.

Now we back to the proof.

Consider the Abelian-by-finite family $X_q \to Y'_q \xrightarrow{\pi} Y$. Recall that there is an equivalence between G_K-sets

$$\pi^{-1}(y_0) \to \{\text{the conjugacy classes of surjections } \pi_1^{\mathrm{geo}}(Y_{\overline{K}}\backslash\{y_0\}, *) \to \mathrm{Aff}(q)\}$$

Through the map $\mathrm{Aff}(q) \to \mathbb{F}_q^*$, we obtain a G_K-map

$$\pi^{-1}(y_0) \to \{\text{the conjugacy classes of surjections } \pi_1^{\mathrm{geo}}(Y_{\overline{K}}\backslash\{y_0\}, *) \to \mathbb{F}_q^* \cong \mathbb{Z}/(q-1)\mathbb{Z}\}$$

The latter set is a subset of $M \triangleq H^1_{\text{ét}}(Y_{\overline{K}}, \mathbb{Z}/(q-1)\mathbb{Z})$ since

$$\pi_1^{\mathrm{geo}}(Y_{\overline{K}}\backslash\{y_0\}, *)^{\mathrm{ab}} \cong \pi_1^{\mathrm{geo}}(Y_{\overline{K}}, *)^{\mathrm{ab}}$$

Let $E \subseteq M$ be the image of $\pi^{-1}(y_0)$. We can easily check that each fiber of $\pi^{-1}(y_0) \to E$ has the same cardinality, and then

$$\mathrm{size}_v(\pi^{-1}(y)(\overline{K})) \leq \mathrm{size}_v(E)$$

Note that M is a $2g$-dimensional free module over $\mathbb{Z}/(q-1)\mathbb{Z}$, and E is the set of all tuples $(y_1, z_1, \cdots, y_g, z_g)$ such that each y_i, z_i is a generator of $\mathbb{Z}/(q-1)\mathbb{Z}$.

A easy calculation shows that

$$\#E = (q-1)^{2g} \prod_{p|q-1} \left(1 - \frac{1}{p^{2g}}\right) \geq \frac{1}{2}(q-1)^{2g}$$

Consider the Weil pairing

$$\langle\cdot,\cdot\rangle : M \times M \to \mu_{q-1}^{\vee}$$

The Frobenius element Frob_v induces an automorphism $T: M \to M$ such that

$$\langle Tv_1, Tv_2 \rangle = q_v^{-1} \langle v_1, v_2 \rangle$$

Note that $\{x \in M : |T \cdot x| < 8\} = \bigcup_{i=1}^{7} \mathrm{Ker}(T^i - 1)$. If $m_1, m_2 \in \mathrm{Ker}(T^i - 1)$ for some $i \leq 7$, then $(q_v^i - 1)\langle m_1, m_2\rangle = 0$. For any odd prime number $r | q-1$, recall that $q_v^i \not\equiv 1 (\mathrm{mod}\ r)$, then $\gcd(q_v^i - 1, q-1) \leq 2$. Thus, $2\langle m_1, m_2\rangle = 0$. The Weil pairing restricting to $2M$ is non-degenerating and vanishing on $2\mathrm{Ker}(T^i - 1)$, then

$$|2\mathrm{Ker}(T^i - 1)| \leq \sqrt{|2M|} = \frac{(q-1)^g}{2^g}$$

Then

$$|\mathrm{Ker}(T^i - 1)| = 2^{2g} \cdot |2\mathrm{Ker}(T^i - 1)| \leq 2^g (q-1)^g$$

Finally

$$\mathrm{size}_v(E) \leq \frac{7 \cdot 2^g (q-1)^g}{\frac{1}{2}(q-1)^{2g}} < \frac{1}{d_q + 1}$$

This completes the proof. ■

5.3 Proof of 5.2.1

Fix $y_0 \in Y(K)^*$, it suffices to prove that $\Omega_v \cap Y(K)^*$ is finite.

We use the results in section 1.1.3. The conditions in 5.2.1 imply that the closure of the image of $\Phi_{\mathbb{C}}$ contains all Lagrangian subspaces of $(H^1_{\mathrm{sing}}(X_z, \mathbb{Q}), \theta)$. Thus, the closure $\overline{\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})}$ is exactly the whole space $\left(\mathrm{Res}_{K_v}^{\kappa(z)_w} L(H^1_{\mathrm{dR}}(X_z/\kappa(z)_w), \theta)\right)(K_v)$, where $L \subseteq \mathcal{F}_{z,w}$ is the Lagrangian Grassmannin parameterized all Lagrangian subspaces. Then $\dim(\overline{\Phi_v(\Omega_v)}) \geq [\kappa(z)_w : K_v]\dfrac{d(d+1)}{2} > 4d^2$ if $[\kappa(z)_w : K_v] \geq 8$. Then (C") holds true. All we need is (B') (and $[\kappa(z)_w : K_v] \geq 8$). It follows from the simplicity lemma below.

Remark 5.3.1. *What we should highlight here is that the Gauss-Manin connection or v-adic parallel transportation we constructed before respects the symplectic structure induced by the polarizations, that is, θ_y coincides with θ_{y_0} via ∇_{y,y_0}. By the similar argument in section 1.1.3, $\nabla_{(z,w),z_0,w_0}$ also respects it.*

Lemma 5.3.2 (Generic simplicity). *There is a finite subset $F \subseteq \Omega_v \cap Y(K)^*$ such that for $y \in \Omega_v \cap Y(K)^* - F$, there exists $(z, w) \in Y'$ above (y, v), where w is a place of $\kappa(z)$ above v, such that:*

(i) $[\kappa(z)_w : K_v] \geq 8$.

(ii) ρ_z *is simple as a* $G_{\kappa(z)}$*-representation.*

Proof of 5.3.2

We call $y \in \Omega_v \cap Y(K)^*$ bad if no $(z, w) \in Y'$ above (y, v) such that the conditions (i), (ii) hold. Take F be the set of bad points.

Lemma 5.3.3. *We will prove the following result: if y is a bad point, then there exists a pair (z, w) above (y, v) satisfying (i), i.e., $[\kappa(z)_w : K_v] \geq 8$, and such that there is a non-zero proper φ-stable subspace $W_{\mathrm{dR}} \subseteq H^1_{\mathrm{dR}}(X_z/\kappa(z)_w) = V_{z,w}$ with*

$$\dim_{\kappa(z)_w} \mathrm{Fil}^1 W_{\mathrm{dR}} \geq \frac{1}{2} \dim_{\kappa(z)_w} W_{\mathrm{dR}}$$

Proof. Assume that there exists y such that we can not choose a such (z,w). For any $z \in \pi^{-1}(y)$, choose W_z as the minimal nonzero sub-representation of ρ_z. For any place $w|v$, there exists a corresponding subspace $W_{z,w} \subseteq H^1_{\mathrm{dR}}(X_z/\kappa(z)_w)$ which is φ-stable and equipped with the corresponding Hodge filtration.

Note that

$$\mathrm{Fil}^2(H^1_{\mathrm{dR}}(X_z/\kappa(z)_w)) = 0$$

By applying 3.3.3 to W_z and v,

$$\sum_{w|v}[\kappa(z)_w : K_v]\frac{\dim \mathrm{Fil}^1 W^{\mathrm{dR}}_{z,w}}{\dim W^{\mathrm{dR}}_{z,w}} = \frac{1}{2}[\kappa(z) : K]$$

then by the assumption

$$\frac{1}{2}\left(\sum_{w|v}[\kappa(z)_w : K_v]\right) = \frac{1}{2}[\kappa(z) : K] \le \sum_{[\kappa(z)_w:K_v]\ge 8}[\kappa(z)_w : K_v]\cdot\frac{d^{\mathrm{dR}}_{z,w}-1}{2d^{\mathrm{dR}}_{z,w}} + \sum_{[\kappa(z)_w:K_v]<8}[\kappa(z)_w : K_v]$$

where $d^{\mathrm{dR}}_{z,w} = \dim(W^{\mathrm{dR}}_{z,w}) \le d = \dim_{\kappa(z)_w}(H^1_{\mathrm{dR}}(X_z/\kappa(z)_w)) = \dim_{\kappa(z)}(H^1_{\mathrm{dR}}(X_z/\kappa(z)))$. Therefore,

$$\frac{1}{2}\left(\sum_{w|v}[\kappa(z)_w : K_v]\right) \le \sum_{[\kappa(z)_w:K_v]\ge 8}[\kappa(z)_w : K_v]\cdot\frac{d-1}{2d} + \sum_{[\kappa(z)_w:K_v]<8}[\kappa(z)_w : K_v]$$

and then

$$\frac{1}{d}\cdot\sum_{[\kappa(z)_w:K_v]\ge 8}[\kappa(z)_w : K_v] \le \sum_{[\kappa(z)_w:K_v]<8}[\kappa(z)_w : K_v]$$

We sum this for all $z \in \pi^{-1}(y)$.

Note that the Frob_v-orbits of $\pi^{-1}(y)(\overline{K})$ are in bijection with pairs (z,w) where z is a Zariski point in $\pi^{-1}(y)$ and $w|v$. Then the orbit containing (z,w) has $[\kappa(z)_w : K_v]$ elements. Let $e_1,\cdots,e_k$ be the cardinality of each Frob_v-orbit of $\pi^{-1}(y)(\bar{K})$. The inequality above means

$$d\cdot\sum_{i:e_i<8} e_i \ge \sum_{e_i\ge 8} e_i$$

which says $\mathrm{size}(\pi^{-1}(y)(\overline{K})) \ge \frac{1}{d+1}$. This contracts the assumption $y \in Y(K)^*$.

Then we can choose (z,w) above (y,v) such that $[\kappa(z)_w : K_v] \ge 8$ and there is a nonzero proper φ-stable subspace $W_{\mathrm{dR}} \le H^1_{\mathrm{dR}}(X_z/\kappa(z)_w)$ with

$$\dim_{\kappa(z)_w}\mathrm{Fil}^1 W_{\mathrm{dR}} \ge \frac{1}{2}\dim_{\kappa(z)_w} W_{\mathrm{dR}}$$

■

If $\mathcal{F}^{\mathrm{bad}}_{z,w}$ is a proper closed subvariety of $\mathcal{F}_{z,w}$, then

$$\dim \mathcal{F}^{\mathrm{bad}}_{z,w} < \frac{d(d+1)}{2}[\kappa(z)_w : K_v]$$

The full monodromy says that the set $\Gamma\cdot h_0^{\mathbb{C}}$ in 1.1.5 contains all the Lagrangian subspace. Then 1.1.5 implies that the Zariski closure of $\Phi_v(\Omega'_v)$ is precisely $\mathcal{F}_v$. Also we have

$$\dim(\mathcal{F}^{\mathrm{bad}}_{z,w}) < \dim(\Gamma\cdot h_0^{\mathbb{C}})$$

Then by 1.1.5 $\Phi_v^{-1}(\mathcal{F}_v^{\mathrm{bad}})$ is contained in a proper K_v-analytic subset of Ω_v'. But $\Omega_v' = \{y \in \mathcal{Y}(\mathcal{O}_v) | y \equiv y_0 (\mathrm{mod}\ v)\} \subseteq \mathcal{Y}(\mathcal{O}_v) = Y(K_v)$ has dimension 1, the result follows immediately.

Hence, the only thing we need to prove is $\mathcal{F}_{z,w}^{\mathrm{bad}}$ is certainly a proper closed subvariety of $\mathcal{F}_{z,w}$. This follows from the following lemma.

Lemma 5.3.4. *Suppose L_w is a finite unramified extension of K_v of degree $r \geq 8$. Let (V, θ) be a symplectic L_w-vector space, with $\dim_{L_w} V = 2d$; let $\varphi : V \to V$ be semilinear for the Frobenius automorphism of L_w/K_v and bijective.*

Then there is a Zariski-open

$$\mathcal{A} \subseteq \mathrm{Res}_{K_v}^{L_w} L(V, \theta)$$

with the following property:

If $F \subseteq V$ is a Lagrangian L_w-subspace, corresponding to a point of $\mathcal{A}(K_v)$, there is no φ-invariant L_w-subspace W of V satisfying

$$\dim(F \cap W) \geq \frac{1}{2} \dim(W)$$

Proof. Similar as 1.1.14, $V \otimes_{K_v} \overline{K_v}$ splits into $2d$-dimension spaces $V_1, \cdots, V_r$ indexed by embeddings $L_w \hookrightarrow \overline{K_v}$, we can extend φ to $\bar{\varphi}$ acting on $V \otimes_{K_v} \overline{K_v}$ such that it identifies all V_i with V_1.

The splits also works for F and W. Then if

$$\dim(F \cap W) \geq \frac{1}{2} \dim(W)$$

we have

$$\dim(F_i \cap W_i) \geq \frac{1}{2} \dim(W_i)$$

Since W is φ-stable, each W_i corresponds to W_1 under the identifications $\bar{\varphi}, \bar{\varphi}^2, \cdots$. Since F is Lagrangian, each F_i is Lagrangian in V_i. The lemma 5.3.5 tells us the set of $(F_1, F_2, \cdots, F_r) \subseteq L(V, \theta)^r$ such that the space W exists forms a proper, Zariski closed subset. ■

Lemma 5.3.5. *Let (V, θ) be a symplectic vector space over a field of characteristic 0 with $\dim(V) = 2d$. Write $L(V, \theta)$ for the Grassmannian of Lagrangian subspaces. Let E be the set of points corresponding to the Lagrangian subspaces*

$$(F_1, F_2, \cdots, F_r) \in L(V, \theta)^r$$

for which there exists a proper nonzero subspace $W \subseteq V$ such that $\dim(F_j \cap W) \geq \frac{1}{2} \dim(W)$ for every j. If $r \geq 8$, then E is a proper, Zariski-closed subset of $L(V, \theta)^r$.

Proof. In fact the result holds for $r \geq 5$.

First we argue that E is Zariski-closed. Consider the product $G(V) \times L(V, \theta)^r$ consisting of the tuples $(W, F_1, F_2, \cdots, F_r)$, where $G(-)$ is the Grassmann. For each i, the dimension $\dim(F_i \cap W)$ is upper semi-continuous. Then the set $\tilde{E} \subseteq G(V) \times L(V, \theta)$ of tuples satisfying the conditions is closed. The image of $\tilde{E}$, which is E, is closed.

Then it suffices to produce a single tuple not in E.

Take $e_1, \cdots, e_d, e_1', \cdots, e_d'$ a standard symplectic basis for V, so $\langle e_i, e_i' \rangle = 1$.

Let

$$F_1 = \mathrm{span}(e_1, e_2, \cdots, e_d)$$

$$F_2 = \mathrm{span}(e'_1, e'_2, \cdots, e'_d)$$

$$F_3 = \mathrm{span}(e_1 + e'_1, e_2 + e'_2, \cdots, e_d + e'_d)$$

$$F_4 = \mathrm{span}(e_1 + e'_1, e_2 + 2e'_2, \cdots, e_d + 2de'_d)$$

Each of these four spaces is maximal isotropic, i.e., Lagrangian, and any two of them have trivial intersection.

Write

$$\pi_{12} : V = F_1 \oplus F_2 \to F_1$$

$$\pi_{21} : V \to F_2$$

Write $\Phi_{12;3} : F_1 \to F_2$ for the isomorphism

$$F_1 \xleftarrow{\pi_{12}^{-1}} F_3 \xrightarrow{\pi_{21}} F_2$$

Explicitly $\Phi_{12;3}$ takes e_i to e'_i. Similarly write $\Phi_{12;4}$ takes e_i to $2ie'_i$.

Assume that W is a proper nonzero subspace W such that $\dim(W \cap F_i) \geq \dfrac{1}{2}\dim(W)$ for $i = 1, 2, 3, 4$. Then $W = (W \cap F_1) \oplus (W \cap F_2)$, and then $W \cap F_1$ and $W \cap F_2$ both have dimension $\dfrac{1}{2}\dim(W)$. Similarly, all $\dim(W \cap F_i) = \dfrac{1}{2}\dim(W)$. The decomposition $W = (W \cap F_1) \oplus (W \cap F_2)$ also says that $\pi_{12}(W) = W \cap F_1$. If we restrict it to $W \cap F_3$, and compare the dimension, we can conclude that

$$\pi_{12} : W \cap F_3 \xrightarrow{\sim} W \cap F_1$$

Similarly

$$\pi_{21} : W \cap F_3 \xrightarrow{\sim} W \cap F_2$$

In particular, $\Phi_{12;3}$ carries $W \cap F_1$ isomorphically to $W \cap F_2$. The same argument holds for $\Phi_{12;4}$. Hence, $W \cap F_1$ is stable under $\Phi_{12;4}^{-1}\Phi_{12;3}$, that is, under $e_i \mapsto 2ie_i$. Then the subspace $W \cap F_1$ only has finitely many possibilities, and there are then only finitely many possibilities for $W \cap F_2$, and then for W. For each W, the condition $\dim(F_5 \cap W) \geq \dfrac{1}{2}\dim(W)$ cuts out a proper Zariski-closed subset of $L(V)$, thus we may choose F_5 not in this set. Then we have constructed a $(F_1, F_2, \cdots, F_5, \cdots) \notin E$. ■

Chapter 6

The proof of 5.2.2

In this chapter, we will construct a nice Abelian-by-finite family satisfying all properties in 5.2.2.

6.1 A result in Hurwitz space

Proposition 6.1.1. Let Y be a curve of genus at least 2 over a number field K. Let G be a center-free finite group. Then there is a smooth projective K-curve Y' with a finite étale morphism $\pi : Y' \to Y$ and a relative curve $Z \to Y'$, satisfying the following properties:

(i) For $y \in Y(\overline{K})$, there is a bijection between $\pi^{-1}(y)(\bar{K})$ and the set of conjugacy classes of surjections $\pi_1^{\mathrm{geo}}(Y - y, *) \to G$ nontrivial on a loop around y. Moreover, if $y \in Y(K)$, then the correspondence is G_K-equivariant.

(ii) There is a finite Y'-morphism $f : Z \to Y' \times Y$, where G acts on Z covering the trivial action on $Y' \times Y$ and making $Z - f^{-1}(\Gamma_\pi) \to Y' \times Y - \Gamma_\pi$ into a G-Galois cover, where $\Gamma_\pi \subseteq Y' \times_K Y$ is the graph of π. For $y' \in Y'(\overline{K})$, the base change $Z_{y'} \to Y$ is branched exactly at $y = \pi(y')$ and the induced morphism $\pi_1^{\mathrm{geo}}(Y - y, *) \to G$ is in the conjugacy class from 1.

Proof. First we assume that Y is a proper smooth curve over $\mathbb{C}$. For $y \in Y(\mathbb{C})$, set $S(y)$ to be the set of conjugacy classes of surjective homomorphisms from $\pi_1^{\mathrm{geo}}(Y - y, *) \twoheadrightarrow G$ (we will omit the index "geo" in this proof), with the property that a loop around y has nontrivial image. Note that this set corresponds to the isomorphism class of connected coverings of Y, with Galois group G, branched only at y.

Set $Y' = \bigsqcup_{y \in Y} S(y)$. Note that Y' is locally simply connected, we can identify all $\pi_1(Y - z, *)$ for $z \in U$, where U is a simply connected neighborhood containing y. Therefore, locally we have a projection $U \times S(y) \to U$. Since Y is connected, we can easily equip Y' with a topology such that $Y' \to Y$ is a covering space. Then Y' has a natural Riemann surface structure with a covering map $\pi : Y' \to Y$. Then, any $z \in Y'$ classifies a connected G-covering of Y branched at $y = \pi(z)$.

Now we take a look at Z. Define $Z_{y'} \to Y$ to be the cover determined by y' branched exactly at $y = e(y')$. Let $Z = \bigsqcup_{z \in Y'} Z_z$. Then there is a natural map $f : Z \to Y' \times Y$. Also, Z has a Riemann surface structure. By the definition of Z, f is a branched cover and G-torsor outside $\Gamma_\pi \subseteq Y' \times Y$. Further, $Z \backslash G = Y' \times Y$.

Now we algebraize everything, that is, we will give Z and Y' structures of complex algebraic varieties compatible with their analytic structures. This follows from [Har13] Appendix B theorem 3.2. Then we can treat all things in the theory of varieties. [Ray06] XII chapter 3 tells us that f is a finite étale morphism away from Γ_π. Further, by composite $Y' \to Y \xrightarrow{e\times \mathrm{id}} Y^2$, the morphism $F: Z \to Y^2$ is etale away from the diagonal Δ. Write $Z^0 = F^{-1}(Y^2 - \Delta)$.

Finally, we want to descend everything from $\mathbb{C}$ to K. In fact, we need a lemma.

Lemma 6.1.2. *(1) The étale cover $F: Z^0 \to Y^2 - \Delta$ can be uniquely extended to a cover $F_K: Z_K^0 \to (Y^2 - \Delta)_K$.*

(2) Let $(y_1, y_0) \in Y(\overline{K})^2$, with $y_1 \neq y_0$. The geometric fiber $F_K^{-1}(y_1, y_0)/G$ is identified with the set $S(y_0)$, as defined before, now using étale $\pi_1^{\mathrm{geo}}(Y - y_0, y_1)$. If $(y_1, y_0) \in Y(K)^2$ this identification is equivariant for G_K.

(3) The quotient Z_K^0/G extends uniquely to an étale cover of Y_K^2. This cover is isomorphic to one of the form $Y_K' \times Y_K \to Y_K^2$ for an étale cover $Y_K' \to Y_K$, such that Y' is the base change of Y_K' to $\mathbb{C}$.

If we assume this lemma, it produces a sequence $Z_K^0 \to Y_K' \times Y_K \to Y_K^2$. Let $Z_K \to Y_K^2$ be the normalization of Y_K^2 inside the fraction field of Z_K^0. Then Z_K is normal and finite over Y_K^2. Then $Z_K \otimes_K \mathbb{C}$ is also finite and normal over Y^2. Consequently, $Z_K \otimes_K \mathbb{C}$ is the normalization of Y^2 in the function field of Z^0. Then the required properties can be checked over $\mathbb{C}$, which is exactly the lemma.

■

Proof of 6.1.2. We first formulate the basic point in purely group theoretic terms.

Let Γ, G be groups, with G finite center-free, and c a conjugacy class of morphisms in $\mathrm{Hom}(\Lambda, \Gamma)$ for some other group Λ. Consider the set $S = S(\Gamma, c, G)$ of all surjective homomorphisms $\varphi: \Gamma \to G$, with the property that they are nontrivial when pulled back by c.

There are natural commuting actions of Γ and G on S:

$$\gamma \cdot \varphi = \varphi \circ \mathrm{Ad}(\gamma)^{-1} (\gamma \in \Gamma),\ \ \varphi \cdot h = \mathrm{Ad}(h^{-1}) \circ \varphi (h \in G)$$

where $\mathrm{Ad}(x): g \mapsto xgx^{-1}$.

This Γ-action extends uniquely to any group $\Gamma' \supseteq \Gamma$ in which Γ is normal and whose conjugation action preserves c. Indeed, the extended action is described by the same formula $\gamma\cdot\varphi = \varphi\circ\mathrm{Ad}(\gamma)^{-1}(\gamma \in \Gamma')$. The uniqueness follows from that

$$\mathrm{Stab}(\varphi) = \{(\gamma, h) \in \Gamma \times G : h\varphi(\gamma x \gamma^{-1})h^{-1},\ \forall x \in \Gamma\} = \{(\gamma, h) \in \Gamma \times G : h^{-1} = \varphi(\gamma)\}$$

the last formula is implied by the surjectivity of φ and center-free property of G.

Now we go back to our question. Fix two points $y_0, y_1 \in Y(\mathbb{C})$. Consider the geometric point $(y_0, y_1) \in Y \times Y$ and the composite of pointed schemes

$$(Y - \{y_0\}, y_1) \xrightarrow{(y,y_1)\mapsto(y,y_0,(y_0,y_1))} (Y^2 - \Delta, (y_0, y_1)) \xrightarrow{(y,y_0,(y_0,y_1))\mapsto(y_0,y_0)} (Y, y_0)$$

Let Γ and Γ' be the geometric étale fundamental group of left two spaces separately.

The long homotopy exact sequence of topological fundamental group implies a short exact sequence of topological fundamental groups, since $\pi_2^{\mathrm{top}}(Y - \{y_0\}) = 1$ and $\pi_0^{\mathrm{top}}(Y, y_0) = 1$. The exactness of the corresponding sequence of geometric étale fundamental groups is obtained by profinite completion, a result in [Sch78] infers this induced sequence is also exact. As a result, Γ is a normal subgroup of Γ'. Then, it is a normal subgroup of the arithmetic fundamental group $\tilde{\Gamma}$ by [Sta23, Tag 0BTX] and the fact that Γ is G_K-invariant. (For H_2 a normal subgroup of H_1, H_1/H_2 acts on the set of normal subgroups of H_2. Then $H_3 \triangleleft H_2$ is a fixed point of this action if and only if it is a normal subgroup of H_1.)

Let c be the conjugacy class of maps $\hat{\mathbb{Z}} \to \pi_1(Y - \{y_0\}, y_1) = \Gamma$ arising from the monodromy around y_0. The commuting $\Gamma \times G$ actions on S define a cover of $Y - \{y_0\}$, equipped with an action of G by automorphisms, whose fiber at y_1 is identified with S. This cover may be described as follows: it is the disjoint union of all the connected G-covers of Y branched precisely at y_0. In other words, it is the restriction of $Z \to Y^2 - \Delta$ to the fiber $\{y_0\} \times (Y - \{y_0\})$. From the uniqueness just described, the extension of this $\Gamma \times G$ action on S to an action of $\tilde{\Gamma}^{\mathrm{geo}} \times G$ corresponds to the cover $Z \to Y^2 - \Delta$. Therefore, the (further) unique extension of the $\Gamma \times G$-action on S to $\tilde{\Gamma} \times G$ gives the statement (1) in the Claim.

Statement (2) follows from the specific (y_1, y_0) chosen above. Statement (3) follows from that the action of Γ on $S(\Gamma, c, G)/G$ is trivial. ■

6.2 The construction

We refer [vdGM07] VI.1 for the basic definitions and properties of Jacobians of curves.

Definition 6.2.1. Given a morphism $\varphi : C_1 \to C_2$ of curves over an algebraically closed field, the associated Prym variety is the cokernel of the induced map $\varphi^* : \mathrm{Pic}^0(C_2) \to \mathrm{Pic}^0(C_1)$ on Jacobians.

The Prym variety is isogenous to the connected component of $\mathrm{Ker}(\mathrm{Pic}^0(C_1) \to \mathrm{Pic}^0(C_2))$, which is a consequence of Poincaré splitting lemma ([vdGM07] Theorem 12.2). Indeed, denote by P the connected component of the kernel, then there exists a complement subvariety Q such that $P \times Q \to \mathrm{Pic}^0(C_1)$ is an isogeny. Recall that the composite $\mathrm{Pic}^0(C_1) \xrightarrow{\varphi} \mathrm{Pic}^0(C_2) \xrightarrow{\varphi^*} \mathrm{Pic}^0(C_1)$ is precisely the multiplication with $\deg(\varphi)$, then Q is isogenous to $\mathrm{Im}(\varphi^*)$.

Now suppose that the covering $C_1 \to C_2$ is Galois, with Galois group $\mathrm{Aff}(q)$, and ramified over exactly one point of C_2. The degree of this covering is $q(q-1)$. Rather than take its Prym variety, however, we prefer to use a reduced version. Namely, we can form a smaller degree-q covering $C_1' \to C_2$ using the permutation action of $\mathrm{Aff}(q)$ on $\mathbb{Z}/q\mathbb{Z}$, and we are interested in the Prym variety of this associated covering, that is,

$$\mathrm{Coker}(\mathrm{Pic}^0(C_2) \to \mathrm{Pic}^0(C_1')))$$

It is easy to see that the image of $\mathrm{Pic}^0(C_2)$ in $\mathrm{Pic}^0(C_1)$ is now the connected component of the $\mathrm{Aff}(q)$-invariants. Similarly, $\mathrm{Pic}^0(C_1')$ is the connected component of the invariants by the subgroup $H_q = (\mathbb{Z}/q\mathbb{Z})^*$, which is a point stabilizer in the permutation action of $\mathrm{Aff}(q)$ on $\mathbb{Z}/q\mathbb{Z}$. Then, $\mathrm{Prym}(C_1'/C_2)$ is isogenous to the cokernel of the map

$$\text{connected component of } \mathrm{Pic}^0(C_1)^{\mathrm{Aff}(q)} \to \text{connected component of } \mathrm{Pic}^0(C_1)^H$$

We may describe all the things above alternatively. Set

$$e = \frac{1}{\#H}\sum_{h\in H} h - \frac{1}{\#\mathrm{Aff}(q)}\sum_{g\in\mathrm{Aff}(q)} g$$

as an idempotent element in $\mathbb{Q}[\mathrm{Aff}(q)]$. Let $e' = 1-e$. Then $e'' = \#\mathrm{Aff}(q)\cdot e'$ acts on $\mathrm{Pic}^0(C_1)$. Moreover, the connected component of $\mathrm{Ker}(e'')$ is isogenous to the Prym variety $\mathrm{Prym}(C_1'/C_2)$.

Then, applying 6.1.1, we have a relative curve $Z_q \to Y_q'$ which equips an $\mathrm{Aff}(q)$-action. Let X_q be the relative identity component of $\mathrm{Pic}^0_{Z_q\to Y_q'}[e'']$ defined above. This X_q is an Abelian scheme over Y_q', equipped with a symmetric and fiberwise ample line bundle. In particular, its fiber over any $y\in Y_q'(\overline{K})$ is isogenous to the reduced Prym variety $\mathrm{Prym}(Z_z^{\mathrm{red}}/Y)$ of the associated $\mathrm{Aff}(q)$-covering $Z_y \to Y$.

6.3 Big monodromy

Definition 6.3.1. For a map $\pi: Z\to Y$ between surfaces, define the primitive homology to be

$$H_1^{\mathrm{Pr}}(Z,Y) = \ker(\pi_*: H_1(Z,\mathbb{Q})\to H_1(Y,\mathbb{Q}))$$

Lemma 6.3.2. *If $\pi: Z\to Y$ is a non-constant morphism of compact Riemann surfaces, then*

$$H_1(\mathrm{Prym}(Z/Y),\mathbb{Q}) = H_1^{\mathrm{Pr}}(Z,Y)$$

Proof. Note that J_Z is isogenous to $J_Y\times \mathrm{Prym}(Z/Y)$, so

$$H_1(J_Z,\mathbb{Q}) = H_1(J_Y,\mathbb{Q})\oplus H_1(\mathrm{Prym}(Z/Y),\mathbb{Q})$$

Choose base point $z_0\in Z$ and $(z_0)\in Y$ and form the corresponding Abel-Jacobian map, then the result follows from the following diagram

$$\begin{tikzcd}
0 \arrow[r] & H_1^{\mathrm{Pr}}(Z,Y) \arrow[r] \arrow[d] & H_1(Z) \arrow[r] \arrow[d, "="] & H_1(Y) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & H_1(\mathrm{Prym}(Z/Y)) \arrow[r] & H_1(J_Z) \arrow[r] & H_1(J_Y) \arrow[r] & 0
\end{tikzcd}$$

■

Now we assume that $\pi: Z\to Y$ is a finitely sheeted topological covering of surfaces. Consider the exact sequence

$$0\to H_1^{\mathrm{Pr}}(Z,Y)\to H_1(Z,\mathbb{Q})\to H_1(Y,\mathbb{Q})\to 0$$

By path lifting, we have a splitting of this sequence $\pi^*: H_1(Y,\mathbb{Q})\to H_1(Z,\mathbb{Q})$, and then we have a decomposition $H_1(Z,\mathbb{Q}) = \pi^*H_1(Y,\mathbb{Q})\oplus H_1^{\mathrm{Pr}}(Z,Y)$. If furthermore the intersection pairing on $H_1(Z,\mathbb{Q})$ is non-degenerate, then it's an orthogonal direct sum, making $H_1^{\mathrm{Pr}}(Z,Y)$ a symplectic $\mathbb{Q}$-vector space.

Definition 6.3.3. A q-sheeted covering $Z\to Y$ between surfaces whose monodromy representation on a general fiber is equivalent to the action of $\mathrm{Aff}(q)$ on $\mathcal{F}_q$ (i.e. we can label the points in the fiber by $\mathcal{F}_q$ such that $\pi_1(Y,y)\to \mathrm{Sym}(\mathcal{F}_q)$ has image $\mathrm{Aff}(q)$) is called an $\mathrm{Aff}(q)$-cover.

Fix a base-point $y_0 \in Y$. For an Aff(q)-cover $\pi : Z \to Y$, its monodromy representation $\pi_1(Y, y_0) \twoheadrightarrow \mathrm{Aff}(q)$ is well-defined up to conjugation by the normalizer of Aff(q) in $\mathrm{Sym}(\mathcal{F}_q)$ (due to various ways to label a fiber in the above definition). But the normalizer is Aff(q) itself. As a result, the isomorphism classes of Aff(q)-covers over Y are in bijection with the set of Aff(q)-conjugacy classes of surjections $\pi_1(Y, y_0) \to \mathrm{Aff}(q)$.

Definition 6.3.4. Let $f : Z \to Y$ be a branched cover between compact Riemann surfaces. If its ramification locus is a singleton $z \in Z$, and $Z^0 \triangleq Z - z \to Y - f(z)$ is an Aff(q)-cover, then $Z \to Y$ is called a singly ramified Aff(q)-cover.

For a singly ramified Aff(q)-cover, Riemann-Hurwitz formula implies that

$$g(Z) = g(Y)q - \frac{q-1}{2}$$

Note that the intersection pairing on $H_1(Z^0, \mathbb{Q}) = H_1(Z)$ is perfect, and we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1^{\mathrm{Pr}}(Z^0, Y-y) & \longrightarrow & H_1(Z-z, \mathbb{Q}) & \longrightarrow & H_1(Y-y, \mathbb{Q}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{=} & & \downarrow{=} & & \\ 0 & \longrightarrow & H_1^{\mathrm{Pr}}(Z, Y) & \longrightarrow & H_1(Z, \mathbb{Q}) & \longrightarrow & H_1(Y, \mathbb{Q}) & \longrightarrow & 0 \end{array}$$

We find that $H_1^{\mathrm{Pr}}(Z^0, Y - y) = H_1^{\mathrm{Pr}}(Z, Y)$.

For a surface Y, recall the natural morphism $\mathrm{MCG}(Y) \to \mathrm{Out}(\pi_1(Y, y_0))$ and that $\mathrm{Out}(\pi_1(Y, y_0))$ acts on the set of Aff(q)-conjugacy classes of surjections $\pi_1(Y, y_0) \to \mathrm{Aff}(q)$, or rather isomorphism classes of Aff(q)-covers over Y.

Given an Aff(q)-cover $\pi : Z \to Y$, let $\mathrm{MCG}(Y)_Z \leq \mathrm{MCG}(Y)$ be the stabilizer of the isomorphism class of π.

By lifting the loop, every $\alpha \in \mathrm{MCG}(Y)_Z$ actually defines an element in $\mathrm{MCG}(Z)$. Thus, there is a natural map $\mathrm{MCG}(Y)_Z \to \mathrm{MCG}(Z)$.

Consider the symplectic representation $\mathrm{MCG}(Z) \to \mathrm{Sp}(H_1(Z, \mathbb{Q}))$ with respect to the intersection pairing. The action on $H_1(Z, \mathbb{Q})$ of an element of $\mathrm{MCG}(Y)_Z$ preserves the decomposition of $H_1(Z, \mathbb{Q})$. If the intersection pairing on $H_1(Z, \mathbb{Q})$ is non-degenerate, then we get a homomorphism

$$\mathrm{Mon} : \mathrm{MCG}(Y)_Z \to \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z, Y))$$

Fix Y a closed surface of genus $g \geq 2$, a base-point $y \in Y$, a prime $q \geq 3$. There are only finitely many isomorphism classes for singly ramified Aff(q)-cover of Y branched at y. Choose a representative system $Z_1, \cdots, Z_N$. Let $\mathrm{MCG}(Y - y)_0 = \bigcap_i \mathrm{MCG}(Y - y)_{Z_i^0}$. Then We have a combined monodromy map $\mathrm{MCG}(Y - y)_0 \to \prod_{i=1}^N \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z_i, Y))$.

Theorem 6.3.5 (Birman exact sequence)**.** Let S be a surface with $\chi(S) < 0$ and $x \in S$. Then the following sequence is exact:

$$1 \to \pi_1(S, x) \to \mathrm{MCG}(S, x) \to \mathrm{MCG}(S) \to 1$$

As $\chi(Y) = 2 - 2g < 0$, we can apply this theorem and we have a morphism $\pi_1(Y, y) \to \mathrm{MCG}(Y - y)$. Let $\pi_1(Y, y)_0 \leq \pi_1(Y, y)$ be the preimage of $\mathrm{MCG}(Y - y)_0$.

Theorem 6.3.6 ([LV20], lemma 8.7)**.** The map $\mathrm{Mon} : \pi_1(Y, y)_0 \to \prod_{i=1}^{N} \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z_i, Y))$ has Zariski-dense image.

By definition, each lift in Y' of a simple closed curve representing a class of $\pi_1(Y, y)_0$ is a simple-closed curve. If we assume this theorem, we can now prove that the Abelian-by-finite family we constrcuted before has full monodromy.

The proof of full monodromy. In the analytic setting, fix $y \in Y$ and for $z \in \pi^{-1}(y)$, the cover $Z_z \to Y$ is singly branched and outside the branch locus is an $\mathrm{Aff}(q)$-Galois cover. The induced homomorphism $\pi_1(Y - y, *) \to G$ maps a loop around y to a q-cycle. The induced morphism $Z_z^{\mathrm{red}} \to Y$ is therefore a singly ramified $\mathrm{Aff}(q)$-cover branched at y, hence isomorphic to a unique $Z_i \to Y$. By the construction of Kodaira-Parshin family, X_z is isogenous to $\mathrm{Prym}(Z_z^{\mathrm{red}}/Y)$, so $\dim X_z = g(Z_z) - 1 = (q-1)(g - \frac{1}{2})$ and $H_1(X_z, \mathbb{Q}) = H_1^{\mathrm{Pr}}(Z_z^{\mathrm{red}}, Y)$. The space $H^1(X_z, \mathbb{Q})$ is dual to $H_1(X_z, \mathbb{Q})$. Since $\{Z_z^{\mathrm{red}} : z \in \pi^{-1}(y)\}$ is in bijection with $\{Z_i : 1 \leq i \leq N\}$, the above theorem shows that the image of the monodromy is full. ■

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致谢